

A solvable model for small- x physics in $D > 4$ dimensions

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Abstract

I present a simplified model for the gluon Green's function governing high-energy QCD dynamics, in arbitrary space-time dimensions. The BFKL integral equation (either with or without running coupling) reduces to a second order differential equation that can be solved in terms of Bessel and hypergeometric functions. Explicit expressions for the gluon density and its anomalous dimension are derived in MS and Q_0 factorization schemes. This analysis illustrates the qualitative features of the QCD gluon density in both factorization schemes. In addition, it clarifies the mathematical properties and validates the results of the “ γ -representation” method [1] proposed by M.Ciafaloni and myself for extracting resummed next-to-leading-log x anomalous dimensions of phenomenological relevance in the two schemes.

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1 Introduction

Small- x resummations in QCD have been extensively investigated in the past years in order to improve the fixed order perturbative description of high-energy hard processes in the small- x regime, where higher order perturbative corrections grow rapidly due to logarithmically enhanced contributions $\sim (\alpha_s \log 1/x)^n$. Knowledge of the precise relationship between the fixed order approach — based on the collinear factorization formula and the DGLAP equation [2] — and the small- x resummed ones — based on the high-energy factorization formula [3] and the BFKL equation [4] — is of course needed for a unified picture of small- x physics, e.g., to provide quantitatively accurate predictions in the small- x region, which will be explored by next-generation colliders.

A major aspect of this relationship is the issue of the factorization scheme employed to define parton densities and coefficient functions. Fixed order perturbative calculations mostly use the (modified) minimal subtraction (MS) scheme in the context of dimensional regularization. On the other hand, small- x resummed approaches — being based on \mathbf{k} -factorization [3] which involves off-shell intermediate particles with non-vanishing transverse momentum \mathbf{k} — are naturally defined in the so-called Q_0 -scheme [5], where infra-red (IR) singularities are regularized by an off-shell probe whose non-vanishing virtuality Q_0^2 plays the role of an IR cutoff.

The basic relations for the $\text{MS} \leftrightarrow Q_0$ scheme change of anomalous dimensions and coefficient functions were obtained some time ago [6, 7] at relative leading-log x (LL x) order, then improved to include next-to-leading-log x (NL x) running coupling corrections [8] and recently extended by M.C. and myself at full NL x level [1]. The main tool of our analysis [1] was the generalization to $4 + 2\epsilon$ dimensions of the γ -representation of the gluon density — a Mellin representation of the BFKL solution in which γ is conjugate to $t \equiv \log(\mathbf{k}^2/\mu^2)$. While for $\epsilon = 0$ the running-coupling BFKL equation is a differential equation in γ , for $\epsilon \neq 0$ it becomes a finite-difference equation, whose solution, however, is not unambiguously determined and has been computed by using sometimes rather formal manipulations. Despite the sensible physical meaning of the procedure and of its results, from a mathematical point of view some steps of our method are not fully proven. It is therefore desirable to have at least an explicit example that could confirm our method of solution of the finite-difference equation, especially in view of its application to compute the anomalous dimensions at full NL x accuracy.

The purpose of the present work is to devise a non-trivial, physically motivated and solvable model which: 1) by providing explicit solutions, illustrates the main qualitative features of the real QCD case; 2) can clarify the less understood aspects of the procedure developed in [1] and verify the correctness of its results. The model I am going to present is a generalization to arbitrary $D = 4 + 2\epsilon$ space-time dimensions of the collinear model [9] used in the past to study the interplay between perturbative and non-perturbative QCD dynamics at high energies. Starting from the formulation of the LL x BFKL equation in D dimensions, only the collinearly enhanced ($\mathbf{k} \gg \mathbf{k}'$ and $\mathbf{k} \ll \mathbf{k}'$) contributions of the integral kernel $K(\mathbf{k}, \mathbf{k}')$ are kept. Despite its poor phenomenological accuracy, this model contains most of the qualitative features of the real theory: it is symmetric in the gluon exchange $\mathbf{k} \leftrightarrow \mathbf{k}'$, it generates collinear singularities in the $\epsilon \rightarrow 0$ limit, it correctly describes the leading-twist LL x behaviour of the gluon density, it includes the running of the coupling. Most importantly, in contrast to the BFKL equation, the collinear model can be solved, as a 1-dimensional Schrödinger-like problem.

Sec. 2 is devoted to the definition of the model in generic number of dimensions. A preliminary study on the qualitative features of the solution of the master integral equation is presented. To this purpose, I briefly review the two types of running-coupling behaviour that are present in $4 + 2\epsilon$ dimensions.

The resolution of the model in the fixed-coupling case is presented in sec. 3. The ensu-

ing integral equation is then recast into a second order differential equation of Bessel type, whose solution provides the unintegrated gluon density. The unintegrated gluon density is first compared with the known perturbative solution [7] and then used to compute the integrated gluon density and anomalous dimension in both the MS-scheme and Q_0 -scheme. The last part of this section concerns the analysis of the Mellin representation of the gluon density and its comparison with the corresponding series and integral representations derived in ref. [1].

Sec. 4 includes the one-loop running coupling. In this case the differential equation is solved in terms of hypergeometric functions. The analyticity properties of the solution will reveal essential in extending the unintegrated gluon density from the IR-free regime — where the coupling is bounded — to the ultra-violet (UV)-free regime — where the Landau pole renders the integral equation meaningless. The explicit results of the b -dependent resummed MS and Q_0 anomalous dimensions — which are shown to agree with the known lowest order running coupling corrections — provide a strong check for the connection between ε -dependence of the kernel and b -dependence of the MS anomalous dimension argued in ref. [1].

A final discussion is reported in sec. 5.

1.1 Notations

I distinguish two symbols of asymptotic behaviour: $f(x) \sim g(x)$ means $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = k$ for some finite and non-zero k , while $f(x) \approx g(x)$ refers to the special case $k = 1$.

The hypergeometric function is denoted by ${}_2F_1(a, b; c; z) \equiv {}_2F_1\left(\begin{matrix} a, & b \\ c & \end{matrix} \middle| z\right)$.

A citation like [1](2.3) means eq. (2.3) of ref. [1].

There are some change of notations between ref. [1] (left side) and this paper (right side):

$$\begin{aligned} \mathcal{F}(\mathbf{k}) &\longrightarrow \omega \mathcal{G}(\mathbf{k}, \mathbf{k}_0) \\ \widetilde{\mathcal{F}}_\varepsilon(\mathbf{k}) e^{\varepsilon \psi(1)} / \Gamma(1 + \varepsilon) &\longrightarrow f_\varepsilon(t) \\ -T &\longrightarrow t_0 . \end{aligned}$$

2 Formulation of the collinear model

In this section, I define a simplified model for the gluon density in high-energy QCD with both running and frozen coupling constant. After recalling the features of the two running coupling regimes, I briefly discuss the expected qualitative behaviour of the solutions of the model.

2.1 Motivation of the model

In high-energy QCD, parton densities and anomalous dimensions are often computed in two different factorization schemes, which differ essentially by the regularization of the infra-red (IR) singularities.

- In the so-called Q_0 scheme [5], the IR regularization occurs by considering off-shell initial partons with non-vanishing virtuality $k^2 = -Q_0^2 < 0$, which plays the role of a momentum cut-off;
- The minimal subtraction (MS) scheme instead, is based on dimensional regularization with on-shell initial partons living in $D = 4 + 2\varepsilon$ space-time dimensions, where IR singularities shows up as poles $\sim 1/\varepsilon^n$ and are subtracted from the physical quantities according to the MS prescription.

The relation between these two schemes can be investigated by including in the defining equations for partons both off-shell initial conditions *and* arbitrary space-time dimensions.

As for the physical case of 4 space-time dimensions, also in generic $D = 4 + 2\varepsilon$ dimensions the high energy (i.e., small- x) behaviour of cross sections in QCD is governed by the gluon Green's function (GGF) $\mathcal{G}_{\omega,\varepsilon}(\mathbf{k}, \mathbf{k}_0)$. Here ω is the Mellin variable conjugated to x , while \mathbf{k} and \mathbf{k}_0 are the transverse momenta of the (reggeized) gluons emerging from the impact-factors of the external particles [3]. The GGF obeys the integral equation (in the following the dependence on the ω variable will always be understood)

$$\omega \mathcal{G}_\varepsilon(\mathbf{k}, \mathbf{k}_0) = \delta^{2+2\varepsilon}(\mathbf{k} - \mathbf{k}_0) + \int \frac{d^{2+2\varepsilon} \mathbf{k}'}{(2\pi)^{2+2\varepsilon}} \mathcal{K}_\varepsilon(\mathbf{k}, \mathbf{k}') \mathcal{G}_\varepsilon(\mathbf{k}', \mathbf{k}_0) \quad (1)$$

where the kernel \mathcal{K}_ε has been determined exactly in the leading-log(x) (LL x) approximation [7] and can be conveniently improved to include subleading corrections (in particular the running of the coupling). Detailed studies of the ensuing solutions and physical consequences has been presented in refs. [7] in the LL x approximation, and in refs. [1, 10] at subleading level.

It should be noted that the NL x approximation limits not only the knowledge of the kernel \mathcal{K}_ε , but also the method of solution of eq. (1). However, it would be desirable to have an exact solution of eq. (1), even with an approximate kernel, in order to understand the overall “non-perturbative” feature of the QCD gluon Green's function. To this purpose, I consider a simplified model for \mathcal{K}_ε whose main virtue is to provide a GGF which can be expressed in terms of known analytic functions. This toy-kernel resembles the field-theoretical one in the collinear regions $\mathbf{k}^2 \ll \mathbf{k}'^2$ and $\mathbf{k}^2 \gg \mathbf{k}'^2$, and has already been considered in the past [9] in order to study the structure of high-energy QCD dynamics in 4 dimensions. In the following I generalize the collinear model to the dimensional regularized theory, with running coupling as well as with fixed coupling constant.

2.2 Definition of the model

The collinear model is defined by the collinear limit $\mathbf{k}_<^2 \ll \mathbf{k}_>^2$ of the LL x BFKL high-energy evolution kernel

$$\mathcal{K}_\varepsilon^{\text{BFKL}}(\mathbf{k}, \mathbf{k}') = \frac{g^2 N_c}{\pi(\mathbf{k} - \mathbf{k}')^2} + \text{virtual terms} \quad \longrightarrow \quad \frac{g^2 N_c}{\pi \mathbf{k}_>^2} \equiv \mathcal{K}_\varepsilon^{\text{coll}}(\mathbf{k}, \mathbf{k}') , \quad (2)$$

$\mathbf{k}_<$ ($\mathbf{k}_>$) being the smallest (biggest) transverse momentum, and g the dimensionful gauge coupling. By introducing the dimensionless coupling constant $\bar{\alpha}_s$, the small- x expansion parameter a and the logarithmic variable t

$$\bar{\alpha}_s \equiv \frac{(g\mu^\varepsilon)^2}{(4\pi)^{1+\varepsilon}\Gamma(1+\varepsilon)} \frac{N_c}{\pi} , \quad a \equiv \frac{\bar{\alpha}_s}{\omega} , \quad t \equiv \log \frac{\mathbf{k}^2}{\mu^2} , \quad (3)$$

we can express both GGF and kernel in terms of the dimensionless quantities f_ε (the unintegrated gluon density) and K^{coll} defined by

$$\omega \mathcal{G}_\varepsilon(\mathbf{k}, \mathbf{k}_0) \equiv \delta^{2+2\varepsilon}(\mathbf{k} - \mathbf{k}_0) + \frac{\Gamma(1+\varepsilon)}{(\pi \mathbf{k}^2)^{1+\varepsilon}} f_\varepsilon(t, t_0) \quad (4)$$

$$\mathbf{k}^2 \mathcal{K}_\varepsilon^{\text{coll}}(\mathbf{k}, \mathbf{k}') \equiv \frac{g^2 N_c}{\pi} K^{\text{coll}}(t - t') . \quad (5)$$

so that one can rewrite eq. (1) in the form¹

$$f_\varepsilon(t, t_0) = a e^{\varepsilon t} \left[K^{\text{coll}}(t - t_0) + \int_{-\infty}^{+\infty} K^{\text{coll}}(t - t') f(t', t_0) dt' \right] \quad (6)$$

$$= a e^{\varepsilon t} \left[\Theta(t_0 - t) e^{t - t_0} + \Theta(t - t_0) + \int_{-\infty}^t f(t', t_0) dt' + \int_t^{+\infty} e^{t - t'} f(t', t_0) dt' \right], \quad (7)$$

where in the second line I have substituted the expression of the collinear kernel

$$K^{\text{coll}}(\tau) = \Theta(-\tau) e^\tau + \Theta(\tau), \quad \tau \equiv t - t' \quad (8)$$

stemming from eqs. (2) and (5).

A second way to relate this model to QCD is to compare the eigenvalue function

$$\chi_{\text{coll}}(\gamma) \equiv \chi(\gamma) \equiv \int_{-\infty}^{+\infty} e^{-\gamma\tau} K^{\text{coll}}(\tau) d\tau = \frac{1}{\gamma} + \frac{1}{1 - \gamma}, \quad (9)$$

with the BFKL one $\chi_{\text{BFKL}} = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$, as in fig. 1. Clearly the two eigenvalue functions display the same qualitative behaviour in the region around and between the leading-twist poles at $\gamma = 0, 1$.

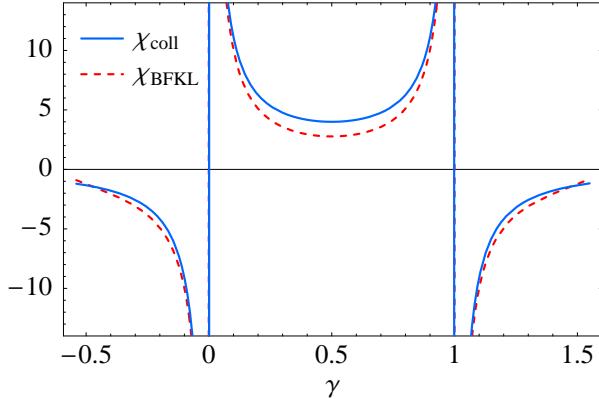


Figure 1: Comparison of the collinear model eigenvalue function (solid-blue) with the BFKL one (dashed-red).

The collinear model can be easily generalized to include the running of the coupling. The small- x parameter a acquires a t -dependence according to the evolution equation

$$\frac{da(t)}{dt} = \varepsilon [a(t) - B a^2(t)], \quad B \equiv \frac{b\omega}{\varepsilon} \quad (10)$$

where b is the one-loop beta function coefficient ($b = 11/12 - N_f/6N_c$ in QCD). The solution of eq. (10) is given by

$$a_t \equiv a(t) = \frac{a e^{\varepsilon t}}{1 + a B (e^{\varepsilon t} - 1)} \quad \iff \quad \left(\frac{1}{a(t)} - B \right)^{-1} = A e^{\varepsilon t}, \quad A \equiv \frac{a}{1 - a B}. \quad (11)$$

Note that in dimensional regularization ($\varepsilon \neq 0$) the coupling $a(t)$ has a non-trivial t -dependence also in the so-called frozen coupling case corresponding to $b = 0$. Substituting $a(t)$ in place

¹In ref. [1] we adopted a step function $\Theta(t + T)$, $T \equiv -t_0$, instead of $K^{\text{coll}}(t - t_0)$ as inhomogeneous term.

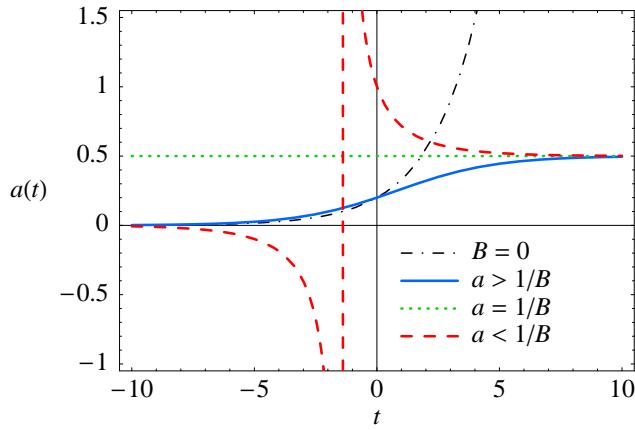


Figure 2: Behaviour of the running coupling $a(t)$ in the regular regime $aB < 1$ (solid-red) and in the Landau regime $aB > 1$ (dashed-blue). The straight line (dotted-green) corresponds to the boundary value $aB = 1$. The case $B = 0$ is represented by the dash-dotted black curve.

of $ae^{\varepsilon t}$ in eq. (6), we obtain, after rearranging some terms, the generalization of the collinear model with running coupling:

$$f_{\varepsilon,b}(t, t_0) = Ae^{\varepsilon t} \left[K^{\text{coll}}(t - t_0) + \int_{-\infty}^{+\infty} K^{\text{coll}}(t - t', B) f_{\varepsilon,b}(t', t_0) dt' \right] \quad (12)$$

$$K^{\text{coll}}(\tau, B) \equiv K^{\text{coll}}(\tau) - B\delta(\tau) . \quad (13)$$

2.3 Running coupling regimes

It is important at this point to realize that the running coupling behaves in two qualitatively different ways, according to whether the parameter aB is greater or less than 1.

- When $aB < 1$, i.e., $\bar{\alpha}_s < \varepsilon/b$, the running coupling $a(t)$ is bounded, positive and increases monotonically from the IR-stable fixed point $a(-\infty) = 0$ to the UV-stable fixed point $a(+\infty) = 1/B$, as shown in fig. 2.
- When $aB > 1$, i.e., $\bar{\alpha}_s > \varepsilon/b$, the running coupling starts from the positive UV-stable fixed point $a(+\infty) = 1/B$, then increases and diverges at the Landau point

$$t_\Lambda \equiv -\frac{1}{\varepsilon} \log(-AB) = \frac{1}{\varepsilon} \log \left(1 - \frac{\varepsilon}{ab} \right) , \quad (14)$$

becomes negative for $t < t_\Lambda$ and finally vanishes at $t = -\infty$. This is the situation realizing the physical limit $\varepsilon \rightarrow 0$ at fixed b .

In the former case, the extra-dimension parameter ε not only regularizes the IR singularities, but avoids also the occurrence of the Landau pole, thus allowing a formulation of the integral equation free of singularities. In practice, the strategy of dimensional regularization consists in computing the physical quantities in the “regular” regime $\bar{\alpha}_s < \varepsilon/b$; the universal ε -singular factors are then removed into non-perturbative quantities, and finally by analytic continuation the physical case at $\varepsilon = 0$ is recovered.

2.4 Qualitative behaviour of the solutions

Before embarking upon the resolution of the collinear model equations (7,12), it is instructive to estimate the qualitative behaviour of the solutions by using well-known methods [5] in the

context of high-energy QCD. Particularly important is the factorization property which allows one to split the unintegrated gluon density $f(t, t_0)$ into a perturbative and a non-perturbative part, provided the “hard scale” $t \gg t_0 \gtrsim t_\Lambda$ is sufficiently large:

$$f(t, t_0) = f_{\text{pt}}(t) f_{\text{np}}(t_0) \times [1 + \mathcal{O}(e^{-t})], \quad (15)$$

up to terms exponentially suppressed in t (higher-twists). In turn, the perturbative factor

$$f_{\text{pt}}(t) \sim \exp \left\{ \int^t \bar{\gamma}(a(t')) dt' \right\} \quad (16)$$

is given in terms of the gluon anomalous dimension $\bar{\gamma}(a(t))$ determined by the small- x equation

$$1 = a(t)\chi(\bar{\gamma}), \quad (17)$$

where χ is the eigenvalue function of the integral kernel in eq. (1).

In this collinear model, the eigenvalue function in eq. (9) provides two solution to eq. (17)

$$\bar{\gamma}_\pm(a) = \frac{1 \pm \sqrt{1 - 4a}}{2}, \quad (18)$$

the perturbative branch being the one with minus sign: $\bar{\gamma}_-(a) = a + \mathcal{O}(a^2)$. At large t , the running coupling saturates at the UV fixed point $a(+\infty) = 1/B = \varepsilon/b\omega$, so that the large- t behaviour of the unintegrated gluon density is given by

$$f(t \gg t_0) \sim \sum_{j=\pm} c_j(t_0) e^{t \bar{\gamma}_j(a(+\infty))} = c_+(t_0) e^{\frac{t}{2}(1 + \sqrt{1 - \frac{4}{B}})} + c_-(t_0) e^{\frac{t}{2}(1 - \sqrt{1 - \frac{4}{B}})}. \quad (19)$$

According to the value of B we expect two kinds of asymptotic behaviour:

- For $B > 4$ the square root is real and positive, the UV regular solution corresponds to the perturbative branch $\bar{\gamma}_-$ of the anomalous dimension and we must reject the (UV irregular) solution which diverges more rapidly: $c_+ = 0$.
- For $B < 4$ the two exponents are complex conjugate, and the gluon density becomes oscillatory at large t . It is not possible to distinguish an UV regular solution, and one has to determine the coefficients c_\pm by analytic continuation in B from $B > 4$. The fixed coupling ($B = 0$) solution belongs to this class.

The above results will be also obtained in a more rigorous way in sec. 4.1, when treating the running-coupling equation.

3 Collinear model with frozen coupling ($b = 0$)

Since the properties of the solution of the collinear model and its connection with the solution method of ref. [1] are more easily illustrated in the fixed coupling case, I start considering the integral equation (7) with $b = 0$.

3.1 Solution in momentum space

The presence of the exponential factor in front of the r.h.s. of eqs. (6,7) spoils scale invariance, therefore the determination of both eigenfunctions and eigenvalues of the integral operator by means of standard techniques is not possible. It turns out, however, that one can exactly solve eq. (7). In fact, by differentiating it twice with respect to t , we obtain the second order differential equation (the ε -dependence of f is understood in this section)

$$f'' - (1 + 2\varepsilon)f' + [\varepsilon(1 + \varepsilon) + a e^{\varepsilon t}]f = -a e^{\varepsilon t_0} \delta(t - t_0) , \quad (20)$$

which can be recast in a more familiar form if we introduce the variables

$$\eta \equiv \frac{1}{\varepsilon} , \quad z \equiv 2\eta\sqrt{ae^{\varepsilon t}} , \quad f(t, t_0) \equiv z^{\eta+2}\mathcal{F}(z, z_0) , \quad (21)$$

thus obtaining

$$z^2\mathcal{F}'' + z\mathcal{F}' + [z^2 - \eta^2]\mathcal{F} = -\frac{\delta(z - z_0)}{2\eta z_0^{\eta-1}} . \quad (22)$$

In the l.h.s. of eq. (22) one recognizes the differential operator defining the Bessel functions $J_{\pm\eta}(z)$ and $Y_\eta(z)$ as solutions of the corresponding homogeneous equation.

The general solution of eq. (22) has the form

$$\mathcal{F}(z, z_0) = c_I(z_0)\mathcal{F}_I(z)\Theta(z_0 - z) + c_U(z_0)\mathcal{F}_U(z)\Theta(z - z_0) \quad (23)$$

where \mathcal{F}_I and \mathcal{F}_U denote respectively the IR-regular and the UV-regular solutions of the homogeneous equation, while c_I and c_U are z_0 -dependent coefficients to be determined by the two conditions of continuity of \mathcal{F} and discontinuity of $\partial_z\mathcal{F}$ at $z = z_0$:

$$\lim_{z \rightarrow z_0^+} \mathcal{F}(z, z_0) - \lim_{z \rightarrow z_0^-} \mathcal{F}(z, z_0) = c_U(z_0)\mathcal{F}_U(z_0) - c_I(z_0)\mathcal{F}_I(z_0) = 0 \quad (24a)$$

$$\lim_{z \rightarrow z_0^+} \partial_z\mathcal{F}(z, z_0) - \lim_{z \rightarrow z_0^-} \partial_z\mathcal{F}(z, z_0) = c_U(z_0)\mathcal{F}'_U(z_0) - c_I(z_0)\mathcal{F}'_I(z_0) = -\frac{1}{2\eta z_0^{\eta+1}} \equiv -N(z_0) . \quad (24b)$$

By solving the above linear system one obtains

$$\mathcal{F}(z, z_0) = \frac{N(z_0)}{W(z_0)} [\mathcal{F}_I(z)\mathcal{F}_U(z_0)\Theta(z_0 - z) + \mathcal{F}_U(z)\mathcal{F}_I(z_0)\Theta(z - z_0)] , \quad (25)$$

where $W = \mathcal{F}_U\mathcal{F}'_I - \mathcal{F}'_U\mathcal{F}_I$ is the Wronskian of the two solutions of the homogeneous equation.

It remains to determine \mathcal{F}_I and \mathcal{F}_U , each being a linear combinations of, say, J_η and Y_η :

$$\mathcal{F}_s(z) = c_s^{(1)}J_\eta(z) + c_s^{(2)}Y_\eta(z) , \quad (s = I, U) \quad (26)$$

(the absolute normalization is irrelevant). From the asymptotic relations

$$J_\eta(z) \sim z^\eta \quad (z \rightarrow 0) \quad J_\eta(z) \sim z^{-1/2} \cos(z + \phi_1) \quad (z \rightarrow +\infty) \quad (27a)$$

$$Y_\eta(z) \sim z^{-\eta} \quad (z \rightarrow 0) \quad Y_\eta(z) \sim z^{-1/2} \cos(z + \phi_2) \quad (z \rightarrow +\infty) \quad (27b)$$

it is clear that the IR-regular solution is $\mathcal{F}_I \propto J_\eta$, since it vanishes more rapidly than any linear combination containing $Y_\eta(z)$ when $z \rightarrow 0$ with $\eta > 0$. On the other hand, the UV-regular solution cannot be determined in this case of $b = 0$, because of the identical asymptotic behaviour (up to normalization and phase) for $z \rightarrow +\infty$ of all solutions in eq. (26). However,

the UV-regular solution can be unambiguously determined in the formulation with running coupling (cf. sec. 4.3), and in the $b \rightarrow 0$ limit it reduces to $\mathcal{F}_U(z) = Y_\eta(z)$. In conclusion

$$\mathcal{F}_I(z) = J_\eta(z) \quad (28)$$

$$\mathcal{F}_U(z) = Y_\eta(z) = [\cos(\pi\eta)J_\eta(z) - J_{-\eta}(z)]/\sin(\pi\eta) \quad (29)$$

$$W(z) = Y_\eta J'_\eta - J_\eta Y'_\eta = -2/\pi z, \quad (30)$$

whence

$$\mathcal{F}(z, z_0) = -\frac{\pi}{4\eta z_0^\eta} [J_\eta(z)Y_\eta(z_0)\Theta(z_0 - z) + Y_\eta(z)J_\eta(z_0)\Theta(z - z_0)]. \quad (31)$$

It is possible to show that $\mathcal{F}(z, z_0)$ in the previous equation obeys also the integral equation (7).

3.2 On-shell limit and perturbative expansion

It is important at this point to check the explicit solution in eq. (31) with known results of the literature. The perturbative expression for the GGF $\mathcal{G}(\mathbf{k}, \mathbf{k}_0)$ in dimensional regularization was given in [7](3.3) for an on-shell ($\mathbf{k}_0 = 0$) initial gluon. In terms of the dimensionless density f_ε their result reads

$$f_\varepsilon(t) = ae^{\varepsilon t} \left[1 + \sum_{m=1}^{\infty} (ae^{\varepsilon t})^m \prod_{j=1}^m \chi(j\varepsilon, \varepsilon) \right] \quad (32)$$

for a generic integral kernel with eigenvalue function $\chi(\gamma, \varepsilon)$.

The on-shell limit $\mathbf{k}_0 \rightarrow 0 \iff t_0 \rightarrow -\infty$ of the unintegrated gluon density $f_\varepsilon(t, t_0)$ at fixed $t, \varepsilon, \bar{\alpha}_s$ is finite, and can be obtained from eqs. (21,31) by exploiting the asymptotic behaviour of Bessel functions $J_\eta(z_0) \approx (z_0/2)^\eta/\Gamma(1 + \eta)$ for $z_0 \rightarrow 0$, whence

$$f_\varepsilon(t) \equiv \lim_{t_0 \rightarrow -\infty} f_\varepsilon(t, t_0) = z^{\eta+2} \lim_{z_0 \rightarrow 0^+} -\frac{\pi}{4\eta z_0^\eta} J_\eta(z_0)Y_\eta(z) = -\frac{\pi}{\eta\Gamma(\eta+1)} \left(\frac{z}{2}\right)^{\eta+2} Y_\eta(z). \quad (33)$$

In words, the on-shell unintegrated gluon density is equal to the UV regular solution of the homogeneous *differential* equation with a proper normalization.

In order to compare the solution (33) with the perturbative expression (32), one has to expand the r.h.s. of eq. (33) in series of $z^2 \sim a$. By rewriting Y_η as a combination of $J_{\pm\eta}$ according to eq. (29), and then using the ascending series [11](9.1.10)

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m!\Gamma(1+\nu+m)}, \quad (34)$$

one obtains

$$\begin{aligned} f_\varepsilon(t) &= \frac{\pi}{\eta\Gamma(\eta+1)\sin(\pi\eta)} \left(\frac{z}{2}\right)^{\eta+2} [J_{-\eta}(z) - \cos(\pi\eta)J_\eta(z)] \\ &= ae^{\varepsilon t} \sum_{m=0}^{\infty} \frac{(-\eta^2 ae^{\varepsilon t})^m}{m!} \frac{\Gamma(1-\eta)}{\Gamma(1-\eta+m)} - \cos(\pi\eta)\Gamma(1-\eta)\eta^{2\eta} (ae^{\varepsilon t})^{\eta+1} \sum_{m=0}^{\infty} \frac{(-\eta^2 ae^{\varepsilon t})^m}{m!\Gamma(1+\eta+m)}. \end{aligned} \quad (35)$$

The first term in the r.h.s. of eq. (35) exactly reproduces the perturbative result (32), since for $m \geq 1$

$$\frac{-\eta^{2m}}{m!} \frac{\Gamma(1-\eta)}{\Gamma(1-\eta+m)} = \prod_{j=1}^m \frac{-1/\varepsilon^2}{j(-\frac{1}{\varepsilon} + j)} = \prod_{j=1}^m \frac{1}{j\varepsilon(1-j\varepsilon)} = \prod_{j=1}^m \chi(j\varepsilon). \quad (36)$$

The second term of eq. (35) provides contributions of order $(ae^{\varepsilon t})^{\eta+1+m} = a^{1+m+1/\varepsilon} e^{[1+(1+m)\varepsilon]t}$, each being outside the domain of the kernel and therefore out of the reach of the iterative procedure. Furthermore, this term is strongly suppressed $\sim a^{\frac{1}{\varepsilon}}$ when $\varepsilon \rightarrow 0$ with respect to the perturbative one. Therefore, it is possible to correctly compute the perturbative coefficients to any order m provided ε is sufficiently small ($\varepsilon < 1/m$). In the limit $\varepsilon \rightarrow 0$ the perturbative solution agrees with the exact one to all orders.²

As final remark, the series in eqs. (32,35) converge for all $t \in \mathbb{R}$, as one can check from the $\gamma \rightarrow +\infty$ asymptotic behaviour of $\chi(\gamma) \sim 1/\gamma$.

3.3 Integrated gluon densities

The major issue this paper is devoted to, concerns the $\text{MS} \leftrightarrow Q_0$ scheme-change, namely the relation between gluon densities and anomalous dimensions in the two factorization schemes. In the collinear model, the off-shell *integrated* gluon density defined by

$$g_\varepsilon(t, t_0) \equiv \int d^{2+2\varepsilon} \mathbf{k}' \omega \mathcal{G}(\mathbf{k}', \mathbf{k}_0) \Theta(\mathbf{k}^2 - \mathbf{k}'^2) = 1 + \int_{-\infty}^t dt' f_\varepsilon(t', t_0) \quad (37)$$

can be computed in closed form (app. A.1), and for $t > t_0$ reads

$$g_\varepsilon(t, t_0) = -\pi \frac{z}{2} \left(\frac{z}{z_0} \right)^\eta J_\eta(z_0) Y_{\eta+1}(z) \quad (t > t_0). \quad (38)$$

Note the remarkable fact that g , like f , is factorized in the t - and t_0 -dependence.

The Q_0 -scheme gluon is given by the $\varepsilon \rightarrow 0$ limit of the above expression, yielding (app. A.2)

$$g^{(Q_0)}(t, t_0) \equiv \lim_{\varepsilon \rightarrow 0} g_\varepsilon(t, t_0) = \frac{a}{\sqrt{1 - 4a \bar{\gamma}(a)}} \exp[\bar{\gamma}(a)(t - t_0)] \quad (39)$$

whence one immediately derives the Q_0 -scheme anomalous dimension (a dot means t -derivative)

$$\gamma^{(Q_0)}(a) \equiv \lim_{t_0 \rightarrow -\infty} \frac{\dot{g}^{(Q_0)}(t, t_0)}{g^{(Q_0)}(t, t_0)} = \bar{\gamma}(a) \quad (\dot{g} \equiv \partial_t g). \quad (40)$$

It is interesting to note that the on-shell limit of the integrated gluon density, i.e., the gluon density in dimensional regularization

$$g_\varepsilon(t) \equiv \lim_{t_0 \rightarrow -\infty} g_\varepsilon(t, t_0) = -\frac{\pi}{\Gamma(\eta+1)} \left(\frac{z}{2} \right)^{\eta+1} Y_{\eta+1}(z) \quad (41)$$

provides the same *effective anomalous dimension* (app. A.2)

$$\gamma_{\text{eff}}(t) \equiv \lim_{\varepsilon \rightarrow 0} \frac{\dot{g}_\varepsilon(t)}{g_\varepsilon(t)} = \lim_{\varepsilon \rightarrow 0} \frac{f_\varepsilon(t)}{g_\varepsilon(t)} = \lim_{\varepsilon \rightarrow 0} \frac{z}{2\eta} \frac{Y_\eta(z)}{Y_{\eta+1}(z)} = \bar{\gamma}(a), \quad (42)$$

which means that the two limiting operations $t_0 \rightarrow -\infty$ and $\varepsilon \rightarrow 0$ commute. Actually, in this model this is a trivial consequence of the factorized structure of the gluon density (38) in its t and t_0 dependence, which causes the ratios \dot{g}/g in eqs. (40) and (42) to be t_0 -independent.

²These conclusions are valid in the off-shell case ($t_0 \in \mathbb{R}$) too, but for sake of simplicity they have been presented only in the on-shell case.

The relation with the *MS-scheme* anomalous dimension is obtained as follows. From the $\varepsilon \rightarrow 0$ asymptotic behaviour of the on-shell gluon density (app. A.2)

$$g_\varepsilon(t) = [1 + \mathcal{O}(\varepsilon)] \frac{a}{\bar{\gamma}(a)[1 - 4a]^{1/4}} \exp \left\{ \frac{1}{\varepsilon} \int_0^{ae^{\varepsilon t}} \frac{da}{a} \bar{\gamma}(a) \right\} \equiv R_\varepsilon(a) g_\varepsilon^{(\text{MS})}(t) \quad (43)$$

one identifies the exponential in eq. (43) as the MS gluon density $g^{(\text{MS})}(t)$,³ since it sums all and only ε -singular terms up to the scale $\mathbf{k}^2 = \mu^2 e^{\varepsilon t}$. The MS anomalous dimension is then computed from the logarithmic derivative

$$\gamma^{(\text{MS})}(a) \equiv \lim_{\varepsilon \rightarrow 0} \frac{\dot{g}_\varepsilon^{(\text{MS})}(t)}{g_\varepsilon^{(\text{MS})}(t)} = \bar{\gamma}(a) \quad (44)$$

and coincides, in this case of $b = 0$, with the Q_0 -scheme anomalous dimension, in agreement with refs. [7] and [1].

The coefficient function R , on the other hand, is finite in the $\varepsilon \rightarrow 0$ limit, and is given by the product [1] $R = \mathcal{N}\mathfrak{R}$, where

$$\mathcal{N}(a) = \frac{1}{\bar{\gamma}(a) \sqrt{-\chi'(\bar{\gamma}(a))}} = \frac{a}{\bar{\gamma}(a)[1 - 4a]^{1/4}} \quad (45)$$

is the fluctuation factor of the saddle-point estimate introduced in [1] (cf. also sec. 3.4), while

$$\mathfrak{R}(a) = \exp \left\{ \int_0^{\bar{\gamma}(a)} \frac{\chi_1(\gamma)}{\chi_0(\gamma)} d\gamma \right\}, \quad \chi(\gamma, \varepsilon) = \chi_0(\gamma) + \varepsilon \chi_1(\gamma) + \mathcal{O}(\varepsilon^2) \quad (46)$$

originates from the ε -dependence of the eigenvalue function. Since in this model χ is independent of ε , $\chi_1 = 0$, $\mathfrak{R} = 1$ and therefore $R = \mathcal{N}$, in agreement with eq. (43).

3.4 Solution in γ space

Having the solution of the integral equation at our disposal, we are ready to check the validity of the procedure suggested in ref. [1], at least in this simplified model. I start reviewing the main steps of that procedure.

1) We introduce for the unintegrated gluon density $f_\varepsilon(t, t_0)$ an integral representation of Mellin-type:

$$f_\varepsilon(t, t_0) = \int_C \frac{d\gamma}{2\pi i} e^{\gamma t} \tilde{f}_\varepsilon(\gamma, t_0). \quad (47)$$

2) In γ -space, the integral equation (7) is thus recast into the finite difference equation

$$\tilde{f}_\varepsilon(\gamma + \varepsilon, t_0) = a\chi(\gamma)e^{-\gamma t_0} + a\chi(\gamma)\tilde{f}_\varepsilon(\gamma, t_0). \quad (48)$$

3) The finite difference equation (48) is solved in terms of a Laurent series in ε , so as to provide the following expression (cf. [1], sec. 2 and eqs. (C.1,C.2)) for the on-shell unintegrated gluon density:

$$f_\varepsilon(t) = \Omega(a, \varepsilon) \int_C \frac{d\gamma}{2\pi i} e^{\gamma t} \exp \left\{ \frac{1}{\varepsilon} \int_0^\gamma L(\gamma') d\gamma' - \frac{1}{2} L(\gamma) + \sum_{n=2}^{\infty} \frac{B_n}{n!} \varepsilon^{n-1} L^{(n-1)}(\gamma) \right\}, \quad (49)$$

³Due to the particular definition of $\bar{\alpha}_s$ in eq. (3) which includes ε -dependent factors, eq. (43) defines a “modified” minimal subtraction scheme, related to the customary MS and $\overline{\text{MS}}$ schemes by a finite scheme change. These details are unimportant for the purpose of this paper.

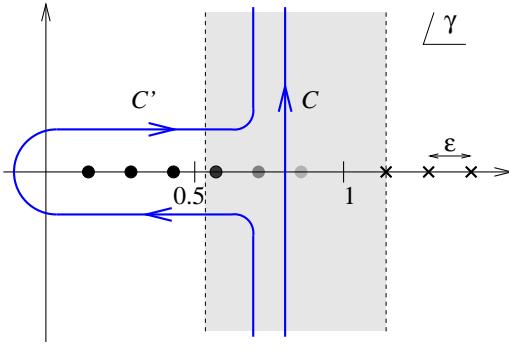


Figure 3: Singularity structure of the Mellin transform $\tilde{f}_\varepsilon(\gamma, t_0)$ in the complex γ -plane. The shaded region corresponds to the convergence strip of the Mellin transform; The crosses indicate the position of the singularities; the circles show the location of the poles of the terms in $\tilde{f}_\varepsilon^{(+)}$; also shown are the original integration path \mathcal{C} in eq. (47), and the deformed contour \mathcal{C}' used in eq. (51).

where $\Omega = \sqrt{a/2\pi\varepsilon}[1 + \mathcal{O}(\varepsilon)]$ is a normalization factor, $L(\gamma) \equiv \log(a\chi(\gamma))$, $L^{(n)} \equiv \partial_\gamma^n L$, and the coefficients B_n denote Bernoulli numbers.

4) The solution is determined by assuming the existence of a saddle point on the real axis, whose steepest descent direction lies on the real axis.

Let us now analyze each point in turn, in the context of the collinear model.

1) Concerning the existence of a Mellin representation for the solution f of the integral equation (7), the asymptotics in eq. (27) guarantee that the Mellin transform \tilde{f} is defined in the strip $1/2 + 3\varepsilon/4 < \Re\gamma < 1 + \varepsilon$ for all $\varepsilon > 0$. Explicitly, \tilde{f} is given in terms of ${}_1F_2(u; u + 1, v; -z_0^2/4)$ sums, as follows:

$$\tilde{f}_\varepsilon(\gamma, t_0) \equiv \left(\int_{-\infty}^{t_0} + \int_{t_0}^{+\infty} \right) dt e^{-\gamma t} f_\varepsilon(t, t_0) \equiv \tilde{f}_\varepsilon^{(-)}(\gamma, t_0) + \tilde{f}_\varepsilon^{(+)}(\gamma, t_0) \quad (50a)$$

$$\tilde{f}_\varepsilon^{(-)}(\gamma, t_0) = \pi \left(\frac{z_0}{2} \right)^\eta Y_\eta(z_0) e^{-\gamma t_0} \sum_{k=1}^{\infty} \frac{(-z_0^2/4)^k}{(k-1)! [k + \eta(1-\gamma)] \Gamma(k+\eta)} \quad (50b)$$

$$\begin{aligned} \tilde{f}_\varepsilon^{(+)}(\gamma, t_0) &= \pi \left(\frac{z_0}{2} \right)^{-\eta} J_\eta(z_0) \left\{ - (a\eta^2)^{\eta\gamma} \cot(\pi\eta\gamma) \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)} \right. \\ &\quad \left. + \frac{e^{-\gamma t_0}}{\sin(\pi\eta)} \sum_{k=1}^{\infty} \frac{(-z_0^2/4)^k}{(k-1)!} \left[\frac{1}{(k-\eta\gamma) \Gamma(k-\eta)} - \frac{(z_0^2/4)^\eta \cos(\pi\eta)}{[k+\eta(1-\gamma)] \Gamma(k+\eta)} \right] \right\}. \end{aligned} \quad (50c)$$

I will show now that, with a proper choice of the contour \mathcal{C} , only the first term of $\tilde{f}_\varepsilon^{(+)}$ in eq. (50c) contributes to the inverse Mellin transform (47) for $t > t_0$ — the relevant region for the on-shell limit. Notice that the analytic continuation of \tilde{f} defines a meromorphic function whose singularities are just the simple poles of $\tilde{f}_\varepsilon^{(-)}$ at $\gamma = 1 + k\varepsilon : k = 1, 2, \dots$, as shown in fig. 3. Actually, $\tilde{f}_\varepsilon^{(+)}$ is holomorphic in the whole plane $\gamma \in \mathbb{C}$, since the poles at $\gamma = k\varepsilon : k = 1, 2, \dots$ stemming from the ratio $\cot(\pi\eta\gamma)/\Gamma(\eta\gamma)$ in the first line of eq. (50c) are exactly canceled by those in the sum on the second line; furthermore, the poles at $\gamma = 1 + k\varepsilon : k = 1, 2, \dots$ stemming from $\Gamma(1+\eta(1-\gamma))$ in the first line are also canceled by those in the sum on the second line.

It is convenient to compute the inverse Mellin transform separately for the $(-)$ and $(+)$ pieces. In the integral of $e^{\gamma t} \tilde{f}_\varepsilon^{(-)}(\gamma, t_0)$ one can close the contour path to the left ($t - t_0 >$

0), without crossing any singularity, thus obtaining a vanishing contribution, as expected. Considering now the integral of $e^{\gamma t} \tilde{f}^{(+)}(\gamma, t_0)$, one is not allowed to close the contour either to the left or to the right, because the factor $e^{\gamma(t-t_0)}$ in front of the sum grows for $\Re(\gamma) \rightarrow +\infty$, while the ratio of gamma-functions in the first term grows with $|\gamma|$ for $\Re(\gamma) < 1/2 + \varepsilon/2$. However, by folding the contour $\mathcal{C} \rightarrow \mathcal{C}'$ so as to let it cross the real axis at some value $\gamma_0 < \varepsilon$ (remember that $\tilde{f}^{(+)}$ has no singularity), and then computing the two contributions of eq. (50c) separately, one obtains a vanishing integral from the second line, because the contour can be closed to the left without crossing any singularities.

To summarize, with an integration contour \mathcal{C}' crossing the real axis at $\gamma_0 < \varepsilon$ and going to infinity with $\Re(\gamma) > 1/2 + \varepsilon/2$ as in fig. 3, only the first term in eq. (50c) contributes in the γ -representation (47) for $t > t_0$.

By performing the on-shell limit I end up with

$$f_\varepsilon(t) \equiv \lim_{t_0 \rightarrow -\infty} f_\varepsilon(t, t_0) = -\frac{\pi}{\Gamma(1+\eta)} \int_{\mathcal{C}'} \frac{d\gamma}{2\pi i} e^{\gamma t} (a\eta^2)^{\eta\gamma} \cot(\pi\eta\gamma) \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)} \equiv \int_{\mathcal{C}'} \frac{d\gamma}{2\pi i} e^{\gamma t} \tilde{f}_\varepsilon(\gamma), \quad (51)$$

which is just the Mellin-Barnes representation [11](9.1.26) of the Bessel function in eq. (33).

2) It is straightforward to check that the on-shell Mellin transform $\tilde{f}_\varepsilon(\gamma)$ in eq. (51) obeys the homogeneous difference equation

$$\tilde{f}_\varepsilon(\gamma + \varepsilon) = a\chi(\gamma) \tilde{f}_\varepsilon(\gamma) \quad (52)$$

analogous to eq. [1](2.11). With some more effort, one can show that the off-shell expression (50) obeys the inhomogeneous difference equation (48).

3) The third issue concerns the validity of eq. (49). By explicitly computing the integral and the derivatives of $L(\gamma)$ in the collinear model

$$\begin{aligned} \int_0^\gamma L(\gamma') d\gamma' &= \gamma \log(a) - \gamma \log(\gamma) + (1-\gamma) \log(1-\gamma) + 2\gamma \\ L^{(m)}(\gamma) &= (m-1)! [(-1)^m \gamma^{-m} + (1-\gamma)^{-m}] \quad (m \geq 1), \end{aligned} \quad (53)$$

the exponent within curly brackets in eq. (49) becomes ($B_{2m+1} = 0 : m \geq 1$)

$$\begin{aligned} S(\gamma) &= \eta [\gamma \log(a) + 2\gamma - \gamma \log(\gamma) + (1-\gamma) \log(1-\gamma)] \\ &\quad - \frac{1}{2} \log(a) + \frac{1}{2} \log(\gamma) + \frac{1}{2} \log(1-\gamma) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \{ [\eta(1-\gamma)]^{1-2m} - [\eta\gamma]^{1-2m} \}. \end{aligned} \quad (54)$$

The sum in the above equation is typical of the asymptotic expansion of the logarithm of the gamma-function [11](6.1.40). In fact, by comparing eq. (54) with the asymptotic expansion

$$\begin{aligned} \log \Gamma(\eta(1-\gamma)) - \log \Gamma(\eta\gamma) &\approx \eta [(1-2\gamma) \log(\eta) - 1 + 2\gamma - \gamma \log(\gamma) + (1-\gamma) \log(1-\gamma)] \\ &\quad + \frac{1}{2} \log(\gamma) - \frac{1}{2} \log(1-\gamma) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \{ [\eta(1-\gamma)]^{1-2m} - [\eta\gamma]^{1-2m} \} \end{aligned} \quad (55)$$

one gets

$$\exp\{S(\gamma)\} \approx \frac{e^\eta}{\sqrt{a} \eta^{\eta+1}} (a\eta^2)^{\eta\gamma} \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)}. \quad (56)$$

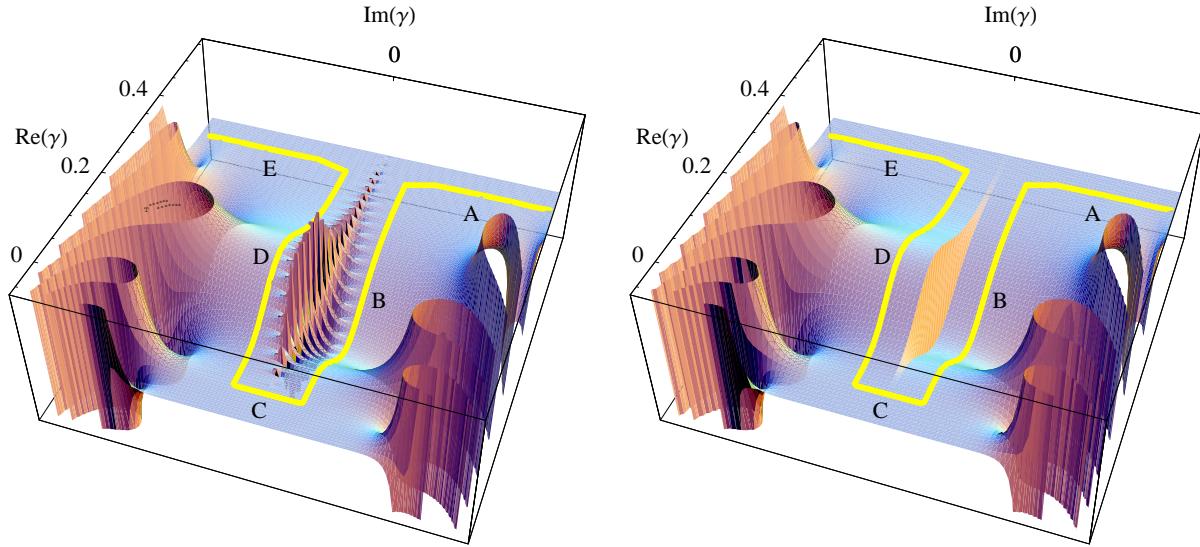


Figure 4: a) The imaginary part of the integrand in eq. (51) showing the singularities on the positive real semi-axis; in yellow a sketch of the fastest convergence path. b) asymptotic limit of the integrand showing the discontinuity (57) on the real axis with a peak around the saddle point value (59).

Apart from the irrelevant normalization factor $e^\eta/\sqrt{a}\eta^{\eta+1}$, eq. (56) agrees with the integrand in eq. (51), when one takes into account that the $\eta \rightarrow +\infty$ asymptotic expansion in powers of $1/\eta$ of $\cot(\pi\eta\gamma)$ is a numeric constant ($\mp i$ according to the sign of $\Im\gamma$).

4) The last step is to evaluate the integral in eq. (51) in the large- η limit. It turns out that, for small values of ε and values of $ae^{\varepsilon t} < 1/\chi(1/2) = 1/4$, the fastest convergence contour path surrounds the interval $0 < \Re(\gamma) < 1/2$ (cf. fig. 4) at a distance decreasing with ε . The main contribution to the integral comes just from this region (parts B and D in fig. 4). In the limit of vanishing ε , the string of poles at $\gamma = k\varepsilon$ accumulates into a branch-cut at $\gamma \in]0, +\infty[$. In fact, while the ratio of gamma-functions is regular at $\gamma > 0$ also in the $\eta \rightarrow +\infty$ limit, the cotangent $\cot(\pi\eta\gamma) \rightarrow -i \operatorname{sign}(\Im\gamma)$ becomes discontinuous across the real axis with a jump equal to $-2i$.

Therefore, neglecting the contributions to the integral in eq. (51) from the parts A, C and E of the contour path, the contributions of B and D amount to the integral in $\gamma \in]0, 1/2[$ of the discontinuity of the integrand, which can be easily obtained by replacing $\cot(\pi\eta\gamma)$ with $-2i$. One obtains

$$f_\varepsilon(t) \approx \frac{1}{\Gamma(\eta+1)} \int_0^{1/2} d\gamma e^{\gamma t} (a\eta^2)^\eta \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)} \approx \frac{\sqrt{a} e^{-\eta}\eta^{\eta+1}}{\Gamma(\eta+1)} \int_0^{1/2} d\gamma e^{\gamma t + S(\gamma)}, \quad (57)$$

where use have been made of eq. (56).

Some remarks are in order. Firstly, by expanding in $\varepsilon \rightarrow 0$ the prefactor in the r.h.s. of eq. (57) $\sqrt{a} e^{-\eta}\eta^{\eta+1}/\Gamma(\eta+1) = \sqrt{a/2\pi\varepsilon}[1 + \mathcal{O}(\varepsilon)]$, eq. (49) is correctly reproduced. Secondly, the integrand, being a discontinuity of a solution of the difference equation (52), is itself a solution of the same equation. Thirdly, the integral representation (57) of the on-shell density f uses an integration path lying on the real axis.

The last remark explains the possibility of having a stable saddle point in the real direction, despite the fact that our original integral in eq. (47) involves a real analytic integrand and an integration contour \mathcal{C} parallel to the imaginary axis. In fact, according to the analysis of [1], in

the $\eta \rightarrow \infty$ limit the leading part of the exponent in eq. (57) is given by

$$\gamma t + S(\gamma) \simeq \gamma t + \eta \int_0^\gamma L(\gamma') d\gamma' + \mathcal{O}(\eta^0) , \quad (58)$$

having considered t a possibly large parameter. The saddle point condition is (cf. eqs. (17,18))

$$\begin{cases} \varepsilon t + \log[a\chi(\bar{\gamma})] = 0 \\ \chi'(\bar{\gamma}) < 0 \end{cases} \iff \bar{\gamma}(ae^{\varepsilon t}) = \frac{1 - \sqrt{1 - 4ae^{\varepsilon t}}}{2} = ae^{\varepsilon t} + \mathcal{O}((ae^{\varepsilon t})^2) \quad (59)$$

and is fulfilled when $4ae^{\varepsilon t} < 1$ so that $0 < \bar{\gamma} < 1/2$. The saddle point behaviour of the discontinuity of the original integrand $\tilde{f}(\gamma)$ is apparent in fig. 4. The final result is

$$f_\varepsilon(t) \approx \frac{1}{\sqrt{-\chi'(\bar{\gamma})}} \exp \left\{ \bar{\gamma} t + \frac{1}{\varepsilon} \int_0^{\bar{\gamma}} \log[a\chi(\gamma')] d\gamma' \right\} = \frac{ae^{\varepsilon t}}{[1 - 4ae^{\varepsilon t}]^{1/4}} \exp \left\{ \frac{1}{\varepsilon} \int_0^{ae^{\varepsilon t}} \frac{da}{a} \bar{\gamma}(a) \right\} , \quad (60)$$

and exhibits the factorization of the $1/\varepsilon$ collinear singularities, thus allowing us to derive the relation between the MS and Q_0 gluons, as explained in the previous section.

In conclusion, the analysis of the frozen-coupling collinear model provides analytic expressions for the gluon densities and anomalous dimensions in both MS- and Q_0 -schemes which agree with the results of ref. [1], sec. 2. In particular, the explicit expression of the Mellin transform $\tilde{f}_\varepsilon(\gamma)$ offers a concrete test of the asymptotic series representation (49), and also allows us to understand the relation between the original Mellin integral (51) and the real-axis integral (57), the latter being the basic tool to prove (by saddle-point estimate) the factorization of collinear singularities (60,43).

4 Collinear model with running coupling ($b > 0$)

In this section I shall extend the collinear model to the more realistic situation of running coupling. I shall show that most of the analysis preformed in sec. 3 for the fixed coupling case can be carried out with running coupling too, with analogous results.

4.1 Asymptotic behaviour of the solutions

Like the $b = 0$ case, the solution of eq. (12) obeys a second order differential equation:

$$(1 + ABe^{\varepsilon t})f'' - (1 + 2\varepsilon + ABe^{\varepsilon t})f' + [\varepsilon(1 + \varepsilon) + Ae^{\varepsilon t}]f = -Ae^{\varepsilon t_0}\delta(t - t_0) . \quad (61)$$

In order to characterize the IR and UV regular solutions of the homogeneous equation, I first determine their large- $|t|$ behaviour. This can be accomplished by rewriting eq. (61) in Schrödinger-like form and then using the WKB approximation. In detail, by letting

$$f(t) \equiv \frac{e^{(\frac{1}{2} + \varepsilon)t}}{1 + ABe^{\varepsilon t}} h(t) , \quad (62)$$

we obtain for h the Schrödinger equation

$$h'' - Vh = 0 , \quad V(t) = \frac{1}{4} - \frac{1}{B} + \frac{1}{B(1 + ABe^{\varepsilon t})} . \quad (63)$$

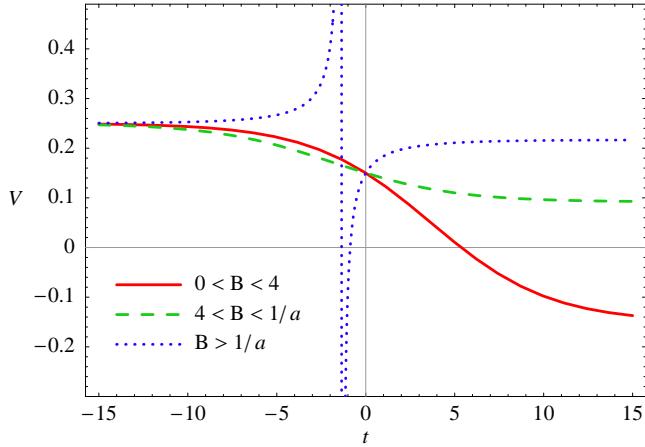


Figure 5: The potential $V(t)$ of the Schrödinger equation (63) in the three running-coupling regimes: $R1$ (solid-red), $R2$ (dashed-green) and $R3$ (dotted-blue).

The WKB approximation to the solution of eq. (63), written in terms of the wave-number $\kappa \equiv \sqrt{V}$, reads

$$h(t) \simeq \frac{1}{\sqrt{\kappa(t)}} \exp \left\{ \pm \int \kappa(t) dt \right\}, \quad (64)$$

and yields, when inserted into eq. (62), the two possible asymptotic behaviours of $f(t)$.

In the IR region ($t \rightarrow -\infty$) we have

$$\lambda \approx e^{(\frac{1}{2}+\varepsilon)t}, \quad V \rightarrow \frac{1}{4}, \quad f \sim \exp \left[\left(\frac{1}{2} + \varepsilon \pm \frac{1}{2} \right) t \right], \quad (65)$$

and one identifies the IR regular solution as the one which vanishes more rapidly, i.e., $f_I \sim e^{(1+\varepsilon)t}$.

In the UV region ($t \rightarrow +\infty$) we have

$$\lambda \sim e^{t/2}, \quad V \rightarrow \frac{1}{4} - \frac{1}{B}, \quad f \sim \exp \left[\frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4}{B}} \right) t \right] \quad (66)$$

and the solutions can have an exponential or oscillatory behaviour according to whether B is greater than or less than 4. In the former case, one again identifies the UV regular solution as the one which vanishes more rapidly, i.e., $f_U \sim \exp \left[\frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{B}} \right) t \right]$.

According to the value of B — and always considering $a < 1/4$ — it is convenient to distinguish 3 regimes where the “potential” V is qualitatively different (cf. fig. 5):

R1: $0 < B < 4$

The potential is regular for all values of $t \in \mathbb{R}$, since $A > 0$, and its UV limit is negative:

$$V(+\infty) = \frac{1}{4} - \frac{1}{B} < 0. \quad (67)$$

As a consequence, in the UV region the wave-number κ is pure imaginary and all solutions share the same oscillatory behaviour, up to a relative phase. The UV regular solution is thus undetermined.

R2: $4 < B < 1/a$

The potential is regular for all values of $t \in \mathbb{R}$, since $A > 0$, and its UV limit $V(+\infty)$ is positive.

Hence, the wave-number κ is real throughout the whole t range and the UV regular solution is uniquely determined, as discussed above.

R3: $1/a < B$

The potential is singular at the Landau point t_Λ of eq. (14), since $A < 0$ and the denominator of V vanishes at $t = t_\Lambda$. This singularity might prevent the existence of a global solution for the integral equation (12). Nevertheless, The UV regular solution of the differential equation (61) can be unambiguously identified.

In the next section I shall explicitly determine the solution of the integral equation (12) in the intermediate regime *R2*, leaving to subsequent sections the analysis of the regime *R1* relevant in the limiting case $b \rightarrow 0$ and of the regime *R3* where the physical situation $\varepsilon \rightarrow 0$ at fixed b is recovered.

4.2 Solution in momentum space

By introducing the new variables

$$\zeta \equiv -ABe^{\varepsilon t} = -\frac{a(t)B}{1-a(t)B}, \quad f(t, t_0) \equiv -\zeta \mathcal{F}(\zeta, \zeta_0) \quad (68)$$

the integral equation (6) becomes

$$\begin{aligned} \mathcal{F}(\zeta, \zeta_0) = & \frac{1}{B(1-\zeta)} \left[\Theta(\zeta_0 - \zeta) + \Theta(\zeta - \zeta_0) \left(\frac{-\zeta}{-\zeta_0} \right)^\eta \right] \\ & - \frac{\eta}{B(1-\zeta)} \left[\int_0^\zeta \mathcal{F}(\zeta', \zeta_0) d\zeta' + (-\zeta)^\eta \int_\zeta^{-\infty} (-\zeta')^{-\eta} \mathcal{F}(\zeta', \zeta_0) d\zeta' \right]. \end{aligned} \quad (69)$$

By differentiating the above equation twice with respect to ζ yields the differential equation

$$\zeta(1-\zeta)\mathcal{F}'' + [(1-\eta) + (\eta-3)\zeta]\mathcal{F}' - \left(\frac{\eta^2}{B} - \eta + 1 \right) \mathcal{F} = \frac{\eta}{B} \delta(\zeta - \zeta_0), \quad (70)$$

whose homogeneous version is just the hypergeometric differential equation with parameters $u, v; w$ given by

$$u, v \equiv 1 - \frac{\eta}{2} \left(1 \pm \sqrt{1 - \frac{4}{B}} \right), \quad w \equiv 1 - \eta = u + v - 1. \quad (71)$$

Let me now consider the regime *R2* in which $4 < B < 1/a$ so that $u < v$ are both real, $A > 0$ and $\zeta < 0$ is a decreasing function of t (cf. eq. (68)). At variance with the $b = 0$ case, both IR ($\zeta \rightarrow 0^-$) and UV ($\zeta \rightarrow -\infty$) regular solutions of the homogeneous differential equation (70), are unambiguously identified, as pointed out in sec. 4.1. Explicitly

$$\mathcal{F}_U(\zeta) \equiv c_U (-\zeta)^{-v} {}_2F_1 \left(\begin{matrix} 2-u, v \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right), \quad c_U \equiv \frac{\Gamma(2-u)\Gamma(1-u)}{\Gamma(1-w)\Gamma(v-u+1)} \quad (72)$$

$$\mathcal{F}_I(\zeta) \equiv (-\zeta)^{1-w} {}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta \right) \quad (73)$$

$$W[\mathcal{F}_U, \mathcal{F}_I] = (w-1)(-\zeta)^{-w}(1-\zeta)^{-2}. \quad (74)$$

By repeating the steps outlined in sec. 3.1, the conditions of continuity of \mathcal{F} at $\zeta = \zeta_0$ and discontinuity of the first derivative $-N(\zeta_0) = \eta/[B\zeta_0(1-\zeta_0)]$ provide the solution of eqs. (69) and (70):

$$\mathcal{F}(\zeta, \zeta_0) = \frac{1-\zeta_0}{B(-\zeta_0)^\eta} [\mathcal{F}_I(\zeta)\mathcal{F}_U(\zeta_0)\Theta(\zeta - \zeta_0) + \mathcal{F}_U(\zeta)\mathcal{F}_I(\zeta_0)\Theta(\zeta_0 - \zeta)], \quad (75)$$

The integrated gluon density defined in eq. (37) can be computed in closed form, and for $t > t_0$ reads (app. A.3)

$$g_{\varepsilon,b}(t, t_0) = c_U \frac{\eta(1 - \zeta_0)}{B(1 - v)(-\zeta_0)^\eta} \mathcal{F}_I(\zeta_0) (-\zeta)^{1-v} {}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right) \quad (t > t_0) . \quad (76)$$

showing also in this case a factorized structure. It is interesting to note that the dependence of the equations and their solutions on t occurs only through ζ , and because of eq. (68), only through the combination $a(t)B$.

The on-shell limit $\zeta_0 \rightarrow 0^-$ of both integrated and unintegrated densities can be easily computed by noting that $\mathcal{F}_I(\zeta_0)/(-\zeta_0)^\eta \rightarrow 1$, whence

$$f_{\varepsilon,b}(t) = \frac{-\zeta}{B} \mathcal{F}_U(\zeta) \quad (77)$$

$$g_{\varepsilon,b}(t) = c_U \frac{\eta}{B(1 - v)} (-\zeta)^{1-v} {}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right) . \quad (78)$$

The comparison of the unintegrated gluon density f with the perturbative solution can be obtained rewriting the UV regular solution \mathcal{F}_U as the sum of two hypergeometric functions with argument ζ by means of the inversion formula [11](15.3.7)

$$\mathcal{F}_U(\zeta) = {}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \middle| \zeta \right) + \frac{\Gamma(2-u)\Gamma(1-u)\Gamma(w-1)}{\Gamma(v)\Gamma(v-1)\Gamma(1-w)} \mathcal{F}_I(\zeta) \quad (79)$$

and then using their series representation [11](15.1.1). The first term yields

$$\begin{aligned} \frac{-\zeta}{B} {}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \middle| \zeta \right) &= A e^{\varepsilon t} \left\{ 1 + \sum_{n=1}^{\infty} (A e^{\varepsilon t})^n \prod_{k=1}^n \frac{(u+k-1)(v+k-1)(-B)}{(w+k-1)k} \right\} \\ &= A e^{\varepsilon t} \left\{ 1 + \sum_{n=1}^{\infty} (A e^{\varepsilon t})^n \prod_{k=1}^n [\chi(k\varepsilon) - B] \right\} , \end{aligned} \quad (80)$$

and provides the perturbative expansion in terms of the parameter $A = a/(1 - aB)$ and of the “effective” eigenvalue function $\chi(\gamma) - B$ relative to the kernel defined in eq. (13). The additional contribution to $f(t)$ due to the second term in eq. (79) is of order $(-\zeta)\mathcal{F}_I \sim (-\zeta)^{2-w} \sim (a e^{\varepsilon t})^{(\eta+1)}$ and cannot find place in the iterative solution, because it does not belong to the domain of the kernel K (cf. sec. 3.2). However, in the $\varepsilon \rightarrow 0$ limit, the perturbative solution (80) agrees with the exact one (77) to all orders.

As last remark, the domain of convergence of the series in eq. (80) is finite ($e^{\varepsilon t} < |AB|^{-1} = |1 - \varepsilon/b\bar{\alpha}_s|$), unlike the $b = 0$ case.

4.3 Fixed coupling limit $b \rightarrow 0$

It is important to study the limit $b \rightarrow 0$ because, as already mentioned in sec. 3.1, it is not possible to determine the UV regular function in the frozen coupling case. Actually, we saw in sec. 4.1 that this problem is also present at $b > 0$ when $B < 4$ (regime *R1*). Therefore, I shall first derive the expression of the UV regular solution \mathcal{F}_U for $B < 4$ and then compute its limiting result at $B = 0$.

Clearly, $B = 4$ is a point of non-analyticity for the coefficients u, v and therefore also for the functions $\mathcal{F}_I, \mathcal{F}_U$. When $B < 4$, the coefficients u, v become complex conjugate. It is easily verified that the IR regular solution \mathcal{F}_I remains real. On the other hand, the expression for

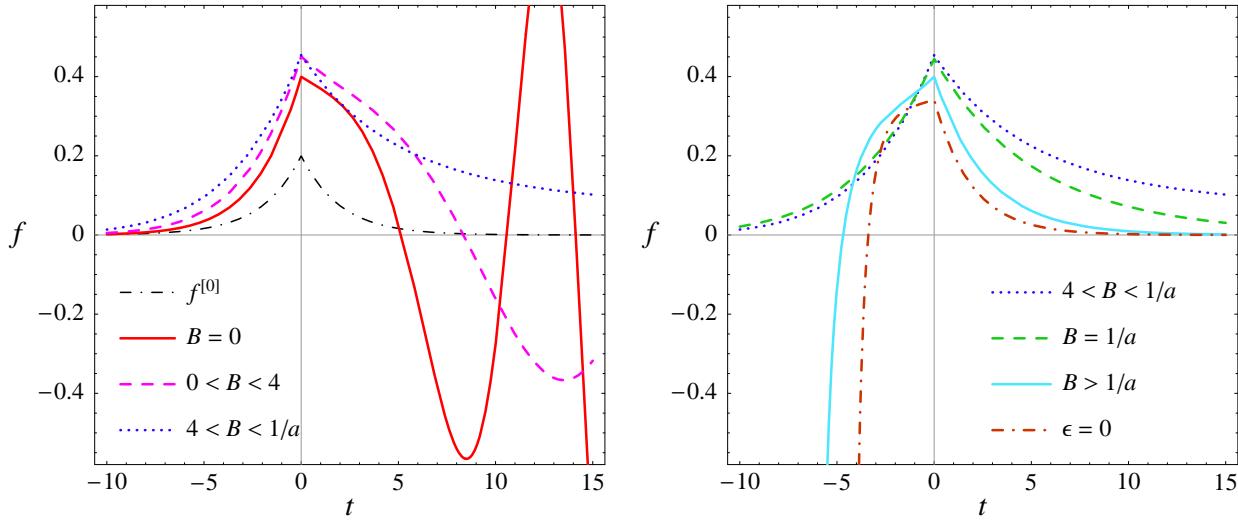


Figure 6: Unintegrated gluon densities (times $e^{-t/2}$) in various running-coupling regimes, with parameters $a = 0.2$, $\eta = 4.5$ and $t_0 = 0$. On the left side: lowest perturbative order (dash-dotted-black), fixed coupling (solid-red), regime R1 (dashed-purple), regime R2 (dotted-blue); on the right side: again regime R2 (dotted-blue), boundary between R2 and R3 (dashed-green), regime R3 (solid-cyan) and $\varepsilon \rightarrow 0$ limit at fixed $b = 1.2$ (dash-dotted-brown). The last two curves diverge at their Landau points.

\mathcal{F}_U in eq. (72) yields two different (complex) results depending on the choice $\Im u = -\Im v$ either greater or less than zero — corresponding to an analytic continuation from $B > 4$ to $B < 4$ in the complex variable $B_{\mathbb{C}}$, from above ($B_{\mathbb{C}} = B + i0$) or from below ($B_{\mathbb{C}} = B - i0$).

Note that $B = 4$ is by no means a critical value for the coefficients of the differential equations (61,70): it only separates the regimes of positive and negative effective potential in the UV region. Since nothing prevents the existence of a real solution, it seems reasonable to define the UV regular solution at $B < 4$ by taking the average of the two analytic continuations:

$$\mathcal{F}_U(\zeta; B < 4) \equiv \frac{1}{2} [\mathcal{F}_U(\zeta; B + i0) + \mathcal{F}_U(\zeta; B - i0)] = \Re \mathcal{F}_U(\zeta; B \pm i0). \quad (81)$$

Of course, the definition (81) joins continuously with the original definition (72) at $B = 4$. An explicit expression of the UV regular solution for all $B > 0$ can be obtained by applying the prescription (81) to eq. (79):

$$\mathcal{F}_U(\zeta) = {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| \zeta\right) + \Re \left(\frac{\Gamma(2-u)\Gamma(1-u)}{\Gamma(v)\Gamma(v-1)} \right) \frac{\Gamma(w-1)}{\Gamma(1-w)} (-\zeta)^{1-w} {}_2F_1\left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta\right) \quad (82)$$

The $B \rightarrow 0$ limit, at fixed a, ε, t is then performed by exploiting the series representation for the hypergeometric functions and the Stirling approximation for the ensuing gamma-functions:

$${}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| \zeta\right) \approx \Gamma(1-\eta) \left(\frac{z}{2}\right)^\eta J_{-\eta}(z) \quad (83a)$$

$${}_2F_1\left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta\right) \approx \Gamma(1+\eta) \left(\frac{z}{2}\right)^{-\eta} J_\eta(z) \quad (83b)$$

$$\Re \frac{\Gamma(2-u)\Gamma(1-u)}{\Gamma(v)\Gamma(v-1)} \approx \Re e^{i\pi\eta} \left(\frac{\eta^2}{B}\right)^\eta = \cos(\pi\eta) \left(\frac{\eta^2}{B}\right)^\eta. \quad (83c)$$

Note in particular the cosine term in eq. (83c) stemming from the real part of the complex exponential: it is exactly the “relative weight” between J_η and $J_{-\eta}$ needed to build up the Bessel function of the second kind Y_η , according to eq. (29). In conclusion, the substitution of the expressions (83) into eq. (75) yields the fixed-coupling result (31), when the proper normalization factors between $\mathcal{F}(z)$ and $\mathcal{F}(\zeta)$ are taken into account.

4.4 Integrated gluon densities

In this section I show how to derive explicit expressions for the integrated gluon densities and anomalous dimensions in the MS-scheme and Q_0 -scheme in the running coupling case. In particular, in this model it is confirmed the claim of ref. [1] that the running coupling corrections to the MS anomalous dimension $\gamma^{(\text{MS})}(a(t), b)$ are provided to all orders by the ε -dependence of the eigenvalue function $\chi(\gamma, \varepsilon)$, according to the relation

$$1 = a(t)\chi(\gamma^{(\text{MS})}, a(t)b\omega), \quad (84)$$

where ε has been replaced by $a(t)b\omega$.

4.4.1 MS-scheme

The MS gluon for $b \neq 0$ is defined in two steps. First, at fixed b and ε , one perturbatively ($\bar{\alpha}_s \rightarrow 0$) computes the integrated gluon in dimensional regularization. This implies that the calculation is naturally performed with $\bar{\alpha}_s < \varepsilon/b$, i.e., $a < 1/B$, which corresponds to the regimes $R1$ and $R2$. Then, all ensuing IR singularities appearing as poles at $\varepsilon = 0$ are isolated and factorized into an IR-singular “transition function”, to be identified with the MS gluon density $g^{(\text{MS})}(t)$.

The important point characterizing the MS-scheme is the factorization of ε -poles in the form

$$g^{(\text{MS})}(t) = \exp \left\{ \int_{-\infty}^t d\tau \gamma^{(\text{MS})}(a(\tau), b) \right\} = \exp \left\{ \frac{1}{\varepsilon} \int_0^{a(t)} \frac{da}{a(1-aB)} \gamma^{(\text{MS})}(a, b) \right\}, \quad (85)$$

where the MS anomalous dimension function $\gamma^{(\text{MS})}$ is required to be ε -independent. The first integral in eq. (85), which is singular for $\varepsilon \rightarrow 0$ because of its IR lower bound, defines the MS anomalous dimension. The second integral is obtained by changing integration variable according to eq. (10), and is more suitable for comparison with perturbative calculations. If eq. (85) contains all IR singularities, the integrated gluon density (78) of the collinear model can be decomposed in the product

$$g_{\varepsilon, b}(t) = R_\varepsilon(a(t), b) \exp \left\{ \frac{1}{\varepsilon} \int_0^{a(t)} \frac{da}{a(1-aB)} \gamma^{(\text{MS})}(a, b) \right\}, \quad (86)$$

where the coefficient function R is regular at $\varepsilon = 0$.

The above expression suggests a method for extracting the MS anomalous dimension. One observes that the integrand in the exponent is singular at $B = 1/a$. On the other hand, for $B \rightarrow 1/a(t)$, no singularity occurs in the off-shell functions $f_{\varepsilon, b}(t, t_0)$ and $g_{\varepsilon, b}(t, t_0)$. This signals that such singularity in the on-shell limit is connected with the infinite evolution of τ from t to $t_0 = -\infty$, and therefore it affects only the exponential factor, while no such singularity is

expected in the coefficient function R . Therefore, if we take the logarithmic derivative of g with respect to $a(t)$ and subsequently the limit $B \rightarrow 1/a(t)$ from below, we obtain ($a_t \equiv a(t)$):

$$\lim_{B \rightarrow 1/a_t} (1 - a_t B) \frac{\partial \log g}{\partial a_t} = \lim_{B \rightarrow 1/a_t} (1 - a_t B) \left[\frac{\partial_{a_t} R}{R} + \frac{\gamma^{(\text{MS})}(a_t, B\varepsilon/\omega)}{\varepsilon a_t (1 - a_t B)} \right] = \frac{\gamma^{(\text{MS})}(a_t, \varepsilon/a_t \omega)}{\varepsilon a_t}. \quad (87)$$

Since the limit can be computed at any $a(t)$ and ε , the above formula enables us to deduce the full functional dependence of $\gamma^{(\text{MS})}$ on both a and b . In this model, from eq. (78) we get

$$\frac{\partial \log g}{\partial a_t} = \frac{\partial \zeta}{\partial a_t} \frac{\partial \log g}{\partial \zeta} = \frac{-B}{(1 - a_t B)^2} \left[\frac{1 - v}{\zeta} - \frac{1}{\zeta^2} \frac{(2 - u)(v - 1)}{v - u + 1} \frac{{}_2F_1 \left(\begin{matrix} 3 - u, v \\ v - u + 2 \end{matrix} \middle| \frac{1}{\zeta} \right)}{{}_2F_1 \left(\begin{matrix} 2 - u, v - 1 \\ v - u + 1 \end{matrix} \middle| \frac{1}{\zeta} \right)} \right], \quad (88)$$

where eq. (68) and the differentiation formula for hypergeometric functions [11](15.2.1) have been used. By performing the limit $B \rightarrow 1/a_t$, which implies $1/\zeta \approx a_t B - 1 \rightarrow 0$, the second term within square brackets does not contribute, and we end up with the simple expression

$$\gamma^{(\text{MS})}(a_t, b) = \varepsilon a_t \lim_{B \rightarrow 1/a_t} (1 - a_t B) \frac{\partial \log g}{\partial a_t} \Big|_{\varepsilon = a_t b \omega} = \varepsilon(1 - v) \Big|_{B=1/a_t, \varepsilon=a_t b \omega} = \frac{1 - \sqrt{1 - 4a_t}}{2}. \quad (89)$$

which coincides with its fixed coupling ($b = 0$) counterpart computed in eqs. (44) and (59). This is a non-trivial result, since it shows that the ε -independent kernel (8) provides a b -independent MS anomalous dimension, according to eq. (84).

Actually, it is not difficult to extend the collinear model to ε -dependent kernels and check eq. (84) in situations where $\gamma^{(\text{MS})}$ is explicitly b -dependent. For instance, by considering an ε -dependent kernel

$$K^{\text{coll}}(\tau, \varepsilon) = \Xi(\varepsilon) [\Theta(-\tau) e^{\xi(\varepsilon)\tau} + \Theta(\tau)], \quad \chi(\gamma, \varepsilon) = \Xi(\varepsilon) \left(\frac{1}{\gamma} + \frac{1}{\xi(\varepsilon) - \gamma} \right), \quad (90)$$

where Ξ and ξ are regular functions of ε , one obtains the same type of differential equation and hypergeometric solutions. Analogous expressions hold for the gluon density, with the replacements $A \rightarrow A\Xi$, $B \rightarrow B/\Xi$ and with new parameters

$$u, v \equiv 1 - \frac{\eta \xi}{2} \left(1 \pm \sqrt{1 - \frac{4\Xi}{B\xi}} \right), \quad w \equiv 1 - \xi \eta. \quad (91)$$

The MS anomalous dimension is then straightforwardly obtained (cf. eq. (89)):

$$\gamma^{(\text{MS})}(a_t, b) = \varepsilon(1 - v) \Big|_{B=1/a_t; \varepsilon=a_t b \omega} = \xi(a_t b \omega) \frac{1 - \sqrt{1 - 4a_t \Xi(a_t b \omega)/\xi(a_t b \omega)}}{2}. \quad (92)$$

On the other hand, the solution $\bar{\gamma}(a_t, \varepsilon)$ of the implicit equation $1 = a_t \chi(\bar{\gamma}, \varepsilon)$ (satisfying the perturbative condition $\bar{\gamma}(0, \varepsilon) = 0$) is given by

$$\bar{\gamma}(a_t, \varepsilon) = \xi(\varepsilon) \frac{1 - \sqrt{1 - 4a_t \Xi(\varepsilon)/\xi(\varepsilon)}}{2} = \varepsilon(1 - v) \Big|_{B=1/a_t} \quad (93)$$

and exactly reproduces the anomalous dimension in eq. (92) when substituting $\varepsilon \rightarrow a_t b \omega$.

4.4.2 Q_0 -scheme

The Q_0 -scheme gluon is defined by the $\varepsilon \rightarrow 0$ limit at fixed b of the off-shell b -dependent integrated density (76), analogously to the frozen-coupling definition in eq. (39). In this limit, $B \rightarrow +\infty$ and we enter the regime $R3$, in which the variable ζ is positive. Therefore, we must extend the expression (76) of the integrated density from $R2$ to $R3$. At the separation point $B = 1/a$, however, the running coupling assumes a constant value $a(t) = a$, whereas the parameter A (eq. (11)) and the variable ζ diverge. In order to avoid the singular point $B = 1/a$, we can change regime by means of an analytic continuation to complex B of the equations (71-78) obtained in $R2$. According to whether $R3$ is reached from the upper or lower half of the $B_C = B \pm i0$ complex plane, $(-\zeta)^p \rightarrow \zeta^p e^{\pm i\pi p}$ acquires a phase of different sign. This translates into a discontinuity of the analytic continuation of g at values of $B > 1/a$. For $t > t_0$ we obtain

$$g_{\varepsilon,b}(t, t_0) = \frac{\eta^2}{B(1-v)(u-v)} \zeta^{1-v} {}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right) g_I(t_0), \quad (t > t_0, B > 1/a) \quad (94)$$

$$g_I(t_0) = (1 - \zeta_0) \left[\zeta_0^{v-2} {}_2F_1 \left(\begin{matrix} u, 2-v \\ u-v+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) - e^{\pm i\pi(u-v)} \frac{\Gamma(2-u)\Gamma(1-u)\Gamma(u-v+1)}{\Gamma(2-v)\Gamma(1-v)\Gamma(v-u+1)} \zeta_0^{u-2} {}_2F_1 \left(\begin{matrix} 2-u, v \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) \right]. \quad (95)$$

From this equation we learn that:

- the analytic continuation of the gluon density is still factorized in its t and t_0 dependence with the same UV (t -dependent) factor as in $R2$;
- the discontinuity affects only the IR factor $g_I(t_0)$, because of the phase $e^{\pm i\pi(u-v)}$ in the second term;
- the Landau pole shows up in the well known branch point of the hypergeometric function at $\zeta = e^{\varepsilon(t-t_\Lambda)} = 1$, in both UV and IR parts.

Our main goal, though, is to obtain the anomalous dimension, which is known to be independent from the IR properties of the theory, provided the “hard scale” t is large enough. In other words, the *effective anomalous dimension*

$$\gamma_{\text{eff}}(t, t_0) \equiv \frac{\dot{g}_{\varepsilon,b}(t, t_0)}{g_{\varepsilon,b}(t, t_0)} = \frac{1-v}{\eta} \frac{{}_2F_1 \left(\begin{matrix} 2-u, v \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right)}{{}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right)}, \quad (\dot{g} \equiv \partial_t g) \quad (96)$$

is expected to depend only on $a(t)$ for $t \gg t_0, t_\Lambda$, where t_Λ eventually represents a cutoff that regularizes the Landau pole and gives mathematical meaning to the gluon density g . In this model, thanks to the factorization property of g , the effective anomalous dimension is independent of t_0 , hence independent on the details of the analytic continuations to $R3$, and needs not a regulator of the Landau pole. As a result, the Q_0 -scheme anomalous dimension

$$\gamma^{(Q_0)}(a(t)) \equiv \lim_{t_0 \rightarrow -\infty} \lim_{\varepsilon \rightarrow 0} \frac{\dot{g}_b(t, t_0)}{g_b(t, t_0)} = \lim_{\varepsilon \rightarrow 0} \gamma_{\text{eff}}(t) \quad (97)$$

is just the $\varepsilon \rightarrow 0$ limit of the effective anomalous dimension.

In order to compute $\gamma^{(Q_0)}$, I rewrite the hypergeometric functions in eq. (96) as [11](15.3.5)

$${}_2F_1 \left(\begin{matrix} 2-u, v-1+n \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right) = \left(1 - \frac{1}{\zeta} \right)^{1-n-v} {}_2F_1 \left(\begin{matrix} v-1, v-1+n \\ v-u+1 \end{matrix} \middle| \frac{1}{1-\zeta} \right), \quad (n=0,1). \quad (98)$$

In this form, only the third parameter and the argument of the hypergeometric function diverge in the $\eta \rightarrow \infty$ limit. Finally, by using the limit representation [12] of the Tricomi confluent hypergeometric function U

$$\lim_{c \rightarrow \infty} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| -\frac{c}{z} \right) = z^a U(a, a-b+1, z), \quad (99)$$

and the four-dimensional running coupling $a(t) = a/(1+ab\omega t) \equiv 1/b\omega T$, I obtain

$$\gamma^{(Q_0)}(a(t), b) = a(t) \frac{U(-1/b\omega, 0, T)}{U(-1/b\omega, 1, T)}, \quad T \equiv t + \frac{1}{ab\omega} \quad (100)$$

which is indeed a function of $a(t)$ and $b\omega$ only. Let me stress that the expression above resums the running coupling corrections of the Q_0 -scheme anomalous dimension to all orders in b .

The Q_0 -anomalous dimension (100) can be directly obtained in $D = 4$ dimensions as well. In fact, the differential equation (61) at $\varepsilon = 0$, in the T variable reduces to the Kummer's equation, whose independent solutions are the confluent hypergeometric functions ${}_1F_1(1-1/b\omega, 2, T)$ and $U(1-1/b\omega, 2, T) = U(-1/b\omega, 0, T)/T$. The UV asymptotic behaviour identifies $U(T)$ as the UV regular solution, i.e., as the unintegrated gluon density $f = \dot{g}$ — up to an IR-dependent normalization constant. In turn, $U(1-1/b\omega, 2, T)$ is the derivative of $b\omega U(-1/b\omega, 1, T)$, which represent therefore the integrated gluon g — up to the same IR-dependent normalization. Finally, the ratio \dot{g}/g yields the Q_0 -anomalous dimension as in eq. (100).

An important check comes from the γ -representation [13]

$$f_{\varepsilon, b}(t) \propto \int \frac{d\gamma}{2\pi i} e^{\gamma t - \frac{X(\gamma)}{b\omega}} = \int \frac{d\gamma}{2\pi i} e^{\gamma t} \gamma^{-1/b\omega} (1-\gamma)^{1/b\omega}, \quad X(\gamma) \equiv \int \chi(\gamma) d\gamma = \log \frac{\gamma}{1-\gamma} \quad (101)$$

yielding, in the collinear model, the confluent hypergeometric function $U(1-1/b\omega, 2, T)$ — apart from a t -independent factor — as first observed by M. Taiuti in her degree thesis [14]. From the representation (101), by means of the saddle-point method, one can obtain the running coupling corrections at any given order in $b\omega$. In the $b \rightarrow 0$ limit, the Q_0 anomalous dimension (100) reduces to the frozen coupling value $\bar{\gamma}(a(t))$ of eq. (40), and coincides with the MS anomalous dimension (89). Starting from $\mathcal{O}(b)$, the two factorization schemes provide different results. In particular, this collinear model predicts a b -independent MS-scheme anomalous dimension, whereas the Q_0 -scheme contains non-vanishing corrections, which agree with the b -expansion in $D = 4$ dimensions.

4.5 Solution in γ space

In this section I show that the γ -representation (47) is valid also at $b \neq 0$ and similar conclusions as for the $b = 0$ case can be drawn in the asymptotic small- ε expansion of the gluon density.

The Mellin transform $\tilde{f}_{\varepsilon, b}(\gamma, t_0)$ of the unintegrated gluon density, defined in eq. (50a), exists in the strip $1/2 - \Re \sqrt{1/4 - 1/B} < \Re \gamma < 1 + \varepsilon$ for all values of ε and B . However, the structure of the singularities of $\tilde{f}(\gamma)$ is different from that in eq. (50), since now $\tilde{f}^{(+)}$ has two infinite series of poles at $\gamma = (1 \pm \sqrt{1-4/B})/2 - \varepsilon n : n \in \mathbb{N}$. In particular, when $B < 4$ (regime R1), these poles are located off the real axis, as shown in fig. 7.

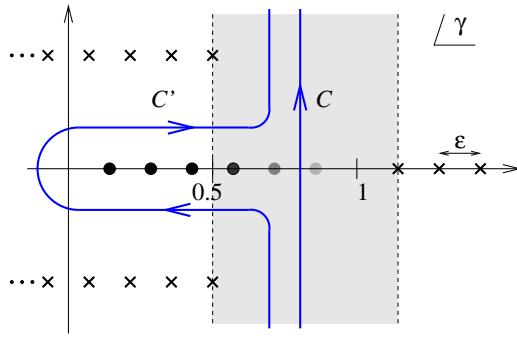


Figure 7: Singularity structure of the Mellin transform $\tilde{f}_\varepsilon(\gamma, t_0)$ in the complex γ -plane. Symbols as in fig. 3.

Let me focus on the gluon density for $t > t_0$. By following the same steps as in sec. 3.4, the function $\tilde{f}^{(+)}(\gamma)$ can be decomposed in the sum of a product of elementary and gamma functions plus hypergeometric functions of type ${}_3F_2$; by then deforming the integration contour \mathcal{C} into \mathcal{C}' so as to cross the real axis at $\gamma_0 < \varepsilon$ (while leaving the new complex poles to the left), the ${}_3F_2$ functions do not contribute to $f_{\varepsilon,b}(t > t_0)$. In the on-shell limit $t_0 \rightarrow -\infty$ we are left with

$$f_{\varepsilon,b}(t) = \frac{\pi\eta}{B\Gamma(\eta)\Gamma(u)\Gamma(v)} \int_{\mathcal{C}} \frac{d\gamma}{2\pi i} e^{\gamma t} (AB)^{\eta\gamma} \Gamma(u-1+\eta\gamma)\Gamma(v-1+\eta\gamma) \times \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)} [-\cot(\pi\eta\gamma) + \Re \cot(\pi u)], \quad (102)$$

where the real part in the last term descends from the real part used in eq. (82). With some trigonometric identities it is not difficult to prove that, for $B < 4$,

$$\Re \cot(\pi u) = \frac{1}{2} [\cot(\pi u) + \cot(\pi v)] = \frac{\sin(\pi\eta)}{\cos(\pi\eta) - \cosh(\pi\eta\sqrt{\frac{4}{B}-1})} \xrightarrow{\eta \rightarrow \infty} \exp\left[-\pi\eta\sqrt{\frac{4}{B}-1}\right]. \quad (103)$$

When computing the inverse Mellin transform (102) along the deformed path \mathcal{C}' in the large- η (fixed B) limit, the term with $-\cot(\pi\eta\gamma) \rightarrow \pm i \text{sign}(\Im\gamma)$ becomes discontinuous on the real axis and its main contribution is provided by the integral of such discontinuity in the real interval $\gamma \in]0, 1/2[$. On the other hand, the term proportional to $\Re \cot(\pi u)$ does not develop any discontinuity, hence does not contribute in the parts B-C-D of the contour; furthermore in the remaining parts A and E it is exponentially suppressed with respect to the other term $\propto \cot(\pi\eta\gamma)$, as indicated in eq. (103), and therefore can be completely neglected. In conclusion

$$f_{\varepsilon,b}(t) \approx \frac{\eta}{B\Gamma(\eta)\Gamma(u)\Gamma(v)} \int_0^{\frac{1}{2}} d\gamma e^{\gamma t} (AB)^{\eta\gamma} \Gamma(u-1+\eta\gamma)\Gamma(v-1+\eta\gamma) \frac{\Gamma(1+\eta(1-\gamma))}{\Gamma(\eta\gamma)}. \quad (104)$$

The main difference between eq. (104) and its $b = 0$ counterpart eq. (57) is the presence of the two additional u, v -dependent gamma-functions. The latter modify the analytic structure of the Mellin transform $\tilde{f}(\gamma)$ away from the real axis, but do not affect the mechanism generating the discontinuity in the $\varepsilon \rightarrow 0$ limit. It is an easy exercise to check that in the $b \neq 0$ case the integrand of eq. (102) (without $e^{\gamma t}$) and its discontinuity (104) obey the homogeneous difference equation

$$\tilde{f}_{\varepsilon,b}(\gamma + \varepsilon) = A\chi(\gamma, B)\tilde{f}_{\varepsilon,b}(\gamma). \quad (105)$$

Finally, in the regime $R1$ we can verify the validity of the Laurent-series representation (49) also for the running coupling case. In fact, by using in eq. (104) the asymptotic expansion of the gamma-functions in terms of Bernoulli numbers, after some calculation one indeed reproduces eq. (49) with

$$\Omega = \sqrt{\frac{A\eta}{2\pi}} \frac{\eta^{\eta+1/2} e^{-\eta}}{\Gamma(\eta+1)} \frac{\Gamma(\eta/B)}{(\eta/B)^{\eta/b-1/2} e^{-\eta/B}}, \quad L(\gamma, B) = \log[A\chi(\gamma, B)]. \quad (106)$$

In the $\eta \rightarrow +\infty$ limit, $\Omega \rightarrow \sqrt{A/2\pi\varepsilon}$ and

$$f_{\varepsilon,b}(t) \approx \frac{1}{\sqrt{2\pi\varepsilon}} \int d\gamma e^{\gamma t} \frac{1}{\sqrt{\chi(\gamma, B)}} \exp \left\{ \frac{1}{\varepsilon} \int_0^\gamma L(\gamma', B) d\gamma' + \frac{\varepsilon}{12} L'(\gamma, B) + \mathcal{O}(\varepsilon^2) \right\} \quad (107)$$

agrees with eq. [1](4.1).

To conclude this section, I have shown that the analysis of the collinear model can be carried out explicitly in the presence of running coupling, and provides analytic results for the anomalous dimensions in both MS- and Q_0 -scheme, which agree with the general results of the literature and in particular with the relations provided by secs. 3 and 4 of ref. [1].

5 Conclusions

In this article I have considered a simplified version of the integral equation that determines the gluon Green's function in high-energy QCD in arbitrary space-time dimensions $D = 4 + 2\varepsilon$. The kernel of the integral equation agrees with the true leading-log x BFKL kernel in the collinear limit, where the transverse momenta of the gluons are strongly ordered. This model has no phenomenological ambition, but embodies most of the qualitative features of the real theory, e.g., the kinematical symmetry in the gluon exchange, the leading-twist behaviour of the gluon density, the pattern of IR singularities and the running coupling. It is therefore a useful tool to check and better understand general results of the QCD literature. In fact, this model was already considered in $D = 4$ dimensions [9] for clarifying the transition mechanism between the perturbative, non-Regge regime and the strong coupling Pomeron behaviour.

In the present formulation I have explicitly determined the gluon densities and their anomalous dimensions in two different factorization schemes: the MS-scheme, based on dimensional regularization, and the Q_0 -scheme, based on an initial off-shell gluon. The main motivation for this analysis stems from a previous work by M.Ciafaloni and myself [1] where we introduced a new method for solving the off-dimensional BFKL equation and for performing the minimal subtraction of the collinear singularities. The rather formal expressions we obtained and some sensible but unproven assumptions we made, could be strongly supported by an explicit non-trivial example where they are shown to be valid.

This analysis attains this object. In fact, in $4 + 2\varepsilon$ dimensions, the master integral equation is solvable in terms of Bessel functions (with frozen coupling) and hypergeometric functions (in the running coupling case). The results obtained here are then often compared with series and integral representations of ref. [1], showing their correctness and their domain of validity. In particular, it is clarified the mechanism by which the integral representation of the solution of the master equation — a real analytic function of the anomalous dimension variable γ integrated along a contour parallel to the imaginary axis — is evaluated by a saddle point integral along the real axis. I also show that the iterative/perturbative solution to the master equation agrees with the exact one, at least up to the order $n < 1/\varepsilon$; higher orders $n > 1/\varepsilon$ of the perturbative expansion do not belong anymore to the domain of the kernel. Among the most important

results there is the confirmation of the formula (84) determining the MS anomalous dimension with running coupling from the ε -dependence of the kernel, which in general was proved only up to $\mathcal{O}(b^2)$ corrections [1].

On the whole, this model represents a useful tool for studying the mathematical properties and the qualitative features of the off-dimensional BFKL equation, even with running coupling. It supports the validity of the procedure [1] for determining anomalous dimensions in subleading approximation, and encourages its application for extracting the leading-twist anomalous dimension at full NLx level.

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A Integrated gluon density

In this appendix I compute the off-shell integrated gluon density defined in eq. (37), both at fixed coupling and with running coupling.

A.1 Integrated gluon with frozen coupling

At fixed coupling, it is convenient to use the z -variable introduced in eq. (21):

$$g_\varepsilon(t, t_0) = 1 + 2\eta \int_0^z x^{\eta+1} \mathcal{F}(x, z_0) \, dx , \quad (108)$$

where the unintegrated density \mathcal{F} is given in eq. (31). For $t < t_0$ we have [11](11.3.20)

$$g_\varepsilon(t < t_0) = 1 + 2\eta \frac{-\pi Y_\eta(z_0)}{4\eta \zeta_0^\eta} \int_0^z x^{\eta+1} J_\eta(x) \, dx = 1 - \frac{\pi}{2} \left(\frac{z}{z_0} \right)^\eta J_{\eta+1}(z) Y_\eta(z_0) . \quad (109)$$

For $t > t_0$, the integral in eq. (108) is conveniently split into 2 pieces

$$\begin{aligned} 2\eta \int_0^z x^{\eta+1} \mathcal{F}(x, z_0) \, dx &= -\frac{\pi}{2z_0^\eta} \left[Y_\eta(z_0) \int_0^{z_0} x^{\eta+1} J_\eta(x) \, dx + J_\eta(z_0) \int_{z_0}^z x^{\eta+1} Y_\eta(x) \, dx \right] \\ &= -\frac{\pi}{2z_0^\eta} \left[z_0^{\eta+1} \{ Y_\eta(z_0) J_{\eta+1}(z_0) - J_\eta(z_0) Y_{\eta+1}(z_0) \} + J_\eta(z_0) z^{\eta+1} Y_{\eta+1}(z) \right] . \end{aligned} \quad (110)$$

The terms in curly brackets are the opposite of the Wronskian (30), [11](9.1.16)

$$J_\eta Y_{\eta+1} - Y_\eta J_{\eta+1} = -W = \frac{2}{\pi z_0} \quad (111)$$

and combined with the prefactors yield a -1 which cancels the $1+$ in the definition (108) of the gluon. The final result reads

$$g_\varepsilon(t > t_0) = -\pi \frac{z}{2} \left(\frac{z}{z_0} \right)^\eta J_\eta(z_0) Y_{\eta+1}(z) \quad (112)$$

and has the remarkable property of being factorized in the z - and z_0 -dependence.

A.2 $\varepsilon \rightarrow 0$ limit

In the $\varepsilon \rightarrow 0$ limit of eq. (112), both the order and the argument of the Bessel functions grow linearly with $\eta \rightarrow +\infty$. By writing the argument in the form

$$z = \eta s, \quad s \equiv 2\sqrt{a}e^{\varepsilon t/2} \rightarrow 2\sqrt{a}, \quad (113)$$

using the asymptotic expansions [11](9.3.6) of the Bessel function J in terms of Airy functions

$$J_\eta(\eta s) \approx \left(\frac{4\zeta}{1-s^2} \right)^{\frac{1}{4}} \eta^{-\frac{1}{3}} \text{Ai}(\eta^{\frac{2}{3}}\zeta), \quad (\eta \rightarrow +\infty) \quad (114)$$

$$\frac{2}{3}\zeta^{\frac{3}{2}} \equiv I(s) \equiv \int_s^1 \frac{\sqrt{1-u^2}}{u} du = \int_{\frac{1}{4}}^{ae^{\varepsilon t}} -\frac{\sqrt{1-4a}}{2} \frac{da}{a} = \log \frac{1+\sqrt{1-s^2}}{s} - \sqrt{1-s^2}, \quad (115)$$

and exploiting the asymptotic expansion [11](10.4.59) of the Airy function

$$\text{Ai}(x) \approx \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right), \quad (x \rightarrow +\infty), \quad (116)$$

one obtains

$$J_\eta(z_0) \approx \frac{\exp[-\eta I(2\sqrt{a}e^{\varepsilon t_0/2})]}{\sqrt{2\pi\eta}(1-4a)^{\frac{1}{4}}}, \quad (117)$$

where I is the integral defined in eq. (115). In the same way, by using the asymptotic expansion [11](9.3.6) of the Bessel function Y in terms of the Airy function $\text{Bi}(\eta^{\frac{2}{3}}\zeta)$ and the large- η expansion of the latter [11](10.4.63), one obtains

$$Y_\eta(\eta s) \approx -\frac{2 \exp[\eta I(s)]}{\sqrt{2\pi\eta}(1-s^2)^{\frac{1}{4}}}. \quad (118)$$

Before applying the above formulas to the integrated gluon, we can immediately derive the $\varepsilon \rightarrow 0$ limit of the unintegrated gluon (cf. eq.(31)) for $t > t_0$. In fact

$$f_\varepsilon(t > t_0) = -\frac{\pi}{\eta} \left(\frac{z}{2}\right)^2 \left(\frac{z}{z_0}\right)^\eta Y_\eta(z) J_\eta(z_0) \approx \frac{a e^{\frac{t-t_0}{2}}}{(1-4a)^{\frac{1}{2}}} \exp\left\{\eta [I(2\sqrt{a}e^{\varepsilon t/2}) - I(2\sqrt{a}e^{\varepsilon t_0/2})]\right\}. \quad (119)$$

The difference of the integrals in the exponential yields

$$I(2\sqrt{a}e^{\varepsilon t/2}) - I(2\sqrt{a}e^{\varepsilon t_0/2}) = -\frac{1}{\varepsilon} \int_{ae^{\varepsilon t_0}}^{ae^{\varepsilon t}} \frac{\sqrt{1-4a}}{2} \frac{da}{a} \xrightarrow{\varepsilon \rightarrow 0} -\frac{\sqrt{1-4a}}{2}(t - t_0), \quad (120)$$

hence

$$f(t > t_0) = \frac{a}{\sqrt{1-4a}} \exp\left[\frac{1-\sqrt{1-4a}}{2}(t-t_0)\right] = \frac{a}{\sqrt{1-4a}} \exp[\bar{\gamma}(a)(t-t_0)], \quad (121)$$

where $\bar{\gamma}(a)$ is the saddle-point value (59) at $\varepsilon = 0$.

As for the integrated gluon density g_ε , comparing eq. (112) with eq. (119) we find

$$\frac{f_\varepsilon(t > t_0)}{g_\varepsilon(t > t_0)} = \frac{z}{2\eta} \frac{Y_\eta(z)}{Y_{\eta+1}(z)}, \quad (122)$$

which is nothing but the effective anomalous dimension in eq. (42). We need the asymptotic behaviour of

$$Y_{\eta+1}(\eta s) = Y_{\tilde{\eta}}((\tilde{\eta} - 1)s) = Y_{\tilde{\eta}}(\tilde{\eta}\tilde{s}) \approx -\frac{2 \exp[\tilde{\eta} I(\tilde{s})]}{\sqrt{2\pi\eta}(1 - \tilde{s}^2)^{\frac{1}{4}}}, \quad (123)$$

where $\tilde{\eta} \equiv \eta + 1$ and $\tilde{s} \equiv s(1 - 1/\tilde{\eta}) \approx s(1 - \varepsilon)$. From

$$\tilde{\eta} I(\tilde{s}) \approx (\eta + 1) \int_{s-\varepsilon s}^1 \frac{\sqrt{1-u^2}}{u} du = (\eta + 1)I(s) + \sqrt{1-s^2} = \eta I(s) + \log \frac{1 + \sqrt{1-s^2}}{s} \quad (124)$$

and $s \rightarrow 2\sqrt{a}$, we obtain a finite limit for the ratio (122)

$$\frac{f_\varepsilon}{g_\varepsilon} \approx \sqrt{a} \exp[\eta I(s) - \tilde{\eta} I(\tilde{s})] \xrightarrow{\varepsilon \rightarrow 0} \left. \frac{\sqrt{a} s}{1 + \sqrt{1-s^2}} \right|_{s=2\sqrt{a}} = \frac{1 - \sqrt{1-4a}}{2} = \bar{\gamma}(a) \quad (125)$$

which coincides with the saddle-point $\bar{\gamma}(a)$ at $\varepsilon = 0$. The explicit expression for the off-shell integrated gluon density at $\varepsilon = 0$ — namely the gluon in the Q_0 -scheme — is finally obtained dividing eq. (121) by $\bar{\gamma}(a)$, whence eq. (39).

In the on-shell case, where the $t_0 \rightarrow -\infty$ limit is performed at non-vanishing ε , the integrated gluon (41) at large- η behaves like

$$\begin{aligned} g_\varepsilon(t) &\approx \frac{a}{\bar{\gamma}(a)(1-4a)^{\frac{1}{4}}} \exp \left\{ \frac{t}{2} + \eta \left[1 + \frac{\log a}{2} + I(s) \right] \right\} \\ &= \mathcal{N}(a) \exp \left\{ \eta \left[1 + \int_1^{ae^{\varepsilon t}} \frac{da}{2a} - \int_{\frac{1}{4}}^{ae^{\varepsilon t}} \frac{\sqrt{1-4a}}{2a} da \right] \right\}. \end{aligned} \quad (126)$$

where I used the Stirling approximation for gamma-functions and eq. (118) in the asymptotic expansion, and an integral representation for the exponent, together with the definition (45) for \mathcal{N} , in the last equality. It is possible to shift the lower limits of integrations to zero, since the two logarithmic singularities at $a = 0$ cancel in the sum of the two integrals. The finite additional contribution is provided exactly by the first term “1+” within square brackets. The final result is

$$g_\varepsilon(t) \approx \mathcal{N}(a) \exp \left\{ \frac{1}{\varepsilon} \int_0^{ae^{\varepsilon t}} \frac{1 - \sqrt{1-4a}}{2} \frac{da}{a} \right\}. \quad (127)$$

Eq. (127) demonstrates the factorization of the collinear singularities, and identifies the ε -finite coefficient factor $R \equiv \mathcal{N}(a)$ and the MS gluon density, according to eq. (43).

A.3 Integrated gluon with running coupling

In the running coupling case, it is convenient to use the ζ -variable introduced in eq. (68). The interesting kinematical region is at $t > t_0$, which in the regimes $R1$ and $R2$ where $aB < 1$ corresponds to $\zeta < \zeta_0 < 0$:

$$g_{\varepsilon,b}(t > t_0) = 1 - \eta \left[\int_0^{\zeta_0} \mathcal{F}(x, z_0) dx + \int_{\zeta_0}^{\zeta} \mathcal{F}(x, z_0) dx \right], \quad (128)$$

where the unintegrated density \mathcal{F} is given in eq. (75).

The first integral involves an integral of hypergeometric function of type [11](15.2.4)

$$\int x^{c-2} {}_2F_1 \left(\begin{matrix} a, b \\ c-1 \end{matrix} \middle| x \right) dx = \frac{x^{c-1}}{c-1} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right), \quad \begin{cases} a = 2-u \\ b = 2-v \\ c = 3-w \end{cases}, \quad (129)$$

where the condition of integrability at $x = 0$ is guaranteed by $c - 2 = \eta > 0$. The second integral in eq. (128), after the position $y = 1/x$, involves an integral of type [11](15.2.3)

$$\int y^{a-1} {}_2F_1 \left(\begin{matrix} a+1, b \\ c \end{matrix} \middle| y \right) = \frac{y^a}{a} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| y \right), \quad \begin{cases} a = v - 1 \\ b = 2 - u \\ c = v - u + 1 \end{cases}. \quad (130)$$

Summing the various contributions yields

$$\begin{aligned} g_{\varepsilon,b}(t > t_0) = 1 - \frac{\eta(1 - \zeta_0)}{B} \frac{\Gamma(2 - u)\Gamma(1 - u)}{\Gamma(1 - w)\Gamma(v - u + 1)} \\ & \times \left\{ (-\zeta_0)^{1-v} \left[\frac{1}{w-2} {}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 3-w \end{matrix} \middle| \zeta_0 \right) {}_2F_1 \left(\begin{matrix} 2-u, v \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) \right. \right. \\ & \quad \left. \left. - \frac{1}{v-1} {}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta_0 \right) {}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) \right] \right. \\ & \quad \left. + \frac{1}{v-1} {}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta_0 \right) (-\zeta)^{1-v} {}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta} \right) \right\}. \end{aligned} \quad (131)$$

The hard task is to prove that the ζ -independent terms, namely those stemming from the square brackets in eq. (131), combine themselves in such a way to give a -1 that cancels the 1 at the beginning of the r.h.s.. The method is to use relations between contiguous hypergeometric functions — differing by one unit in some of their (a, b, c) parameters — and their derivatives.

By introducing the short-hand notation

$$F_1 \equiv {}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 2-w \end{matrix} \middle| \zeta_0 \right), \quad F_2 \equiv {}_2F_1 \left(\begin{matrix} 2-u, v \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) \quad (132)$$

and exploiting the relations [11](15.2.6) and [11](15.2.5), one gets

$${}_2F_1 \left(\begin{matrix} 2-u, 2-v \\ 3-w \end{matrix} \middle| \zeta_0 \right) = \frac{2-w}{(1-u)(1-v)} [(1 - \zeta_0)F'_1 - F_1] \quad (133)$$

$${}_2F_1 \left(\begin{matrix} 2-u, v-1 \\ v-u+1 \end{matrix} \middle| \frac{1}{\zeta_0} \right) = \left(1 - \frac{2-u}{\zeta_0(1-u)} \right) F_2 + \frac{\zeta_0 - 1}{\zeta_0^2(1-u)} F'_2, \quad (134)$$

whence

$$[\dots]_{(131)} = \frac{1}{(1-u)(1-v)} \left\{ (2-u) \left(1 - \frac{1}{\zeta_0} \right) F_1 F_2 + (\zeta_0 - 1) \left(F'_1 F_2 + \frac{1}{\zeta_0^2} F_1 F'_2 \right) \right\}. \quad (135)$$

In order to find a relation among the F_j 's and their derivatives, I exploit the Wronskian (74):

$$\frac{W[\mathcal{F}_U, \mathcal{F}_I](\zeta_0)}{c_U} = (-\zeta_0)^{1-v-w} \left\{ \left(\frac{1+v-w}{\zeta_0} \right) F_1 F_2 + F'_1 F_2 + \frac{1}{\zeta_0^2} F_1 F'_2 \right\}. \quad (136)$$

The combination $F'_1 F_2 + F_1 F'_2 / \zeta_0^2$ entering eq. (135) can thus be expressed in terms of the product $F_1 F_2$. As a result, the terms with $F_1 F_2$ cancel out and all gamma-functions simplifies. Finally, by substituting the explicit expressions (71) of u, v, w , it is straightforward to compute the sum of the ζ -independent terms in eq. (131) and to obtain -1 , as I stated previously. The remaining ζ -dependent term provides the factorized expression (76) for the integrated gluon density.

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